V.P. & R.P.T.P. Science College, V.V.Nagar

Internal Test: 2018-19

Subject : Mathematics | US05CMTH03 | Max. Marks : 50

Metric Spaces

Date: 03/10/2017 | Timing: 10.00 am - 12.00 am

Q: 1. Answer the following by choosing correct answers from given choices. 8

[1] Every Cauchy sequence is
[A] convergent [B] is not always convergent [C] divergent [D] none

[2] Every function defined on R_d is continuous at
[A] all real numbers [B] 0 and 1 only
[C] rational numbers only [D] irrational numbers only

[3] In the metric space $M = [0,1]$ with usual metric , $B[\frac{1}{4}, 1] =$
[A] $[0,1]$ [B] $[\frac{1}{4}, 1]$ [C] $[0, \frac{1}{4}]$ [D] $(0,1)$

[4] In a metric space (M, ρ), its subsets M and ϕ are
[A] open but not closed [B] closed but not open
[C] open as well as closed [D] neither open nor closed

[5] The subset ______ of R^1 is not complete
[A] $[1,2]$ [B] $(0,1)$ [C] $[0,2]$ [D] R

[6] Every singletone subset of _____ is open
[A] R_d [B] R^1 [C] R^2 [D] $[0,1]$, (with usual metric)

[7] The range of a continuous function f defined on $[1,2]$ is
[A] unbounded [B] compact [C] disconnected [D] none

[8] Every finite subset of a metric space is
[A] unbounded [B] compact [C] dense [D] none

Q: 2. Answer any FIVE of the following. 10

[1] For $P(x_1, y_1)$ and $Q(x_2, y_2)$ in $\mathbb{R}^2$, define $\sigma : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$\sigma(P,Q) = |x_1 - x_2| + |y_1 - y_2|$$

, show that σ is a metric on $\mathbb{R}^2$

[2] Let $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $d(x,y) = \sin|x - y|$. Check whether d is a metric or not.

[3] For the discrete metric $\mathbb{R}_d$, find (1) $B[a; 2]$ (2) $B[a; 1/2]$

[4] Prove that in any metric space (M, ρ), both M and ϕ are open sets.

[5] Show that every subset of $\mathbb{R}_d$ is bounded

[6] Define : (i) Totally bounded set (i) ϵ-dense set

[7] Show that a finite subset of $\mathbb{R}_d$ is compact.

[8] Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is uniformly continuous.

Q: 3 [A] prove that a real valued function f is continuous at $a \in R$ iff whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers converging to a then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$. 5

[B] Show that a sequence in a metric space cannot convergene to two distinct limits. 3

OR

Q: 3 [A] In usual notations prove that $\lim_{x \longrightarrow a} [f(x).g(x)] = \lim_{x \to a} f(x). \lim_{x \to a} g(x)$ 5

[B] Prove that a real valued function f is continuous at $a \in R^1$ if and only if the inverse image under f of any open ball $B[f(a), \epsilon]$ about $f(a)$ contains an open ball $B[a, \delta]$ about a 3

Q: 4 [A] Prove that if (M, ρ) is a metric space then any open sphere in M is an open set. 5

[B] Let $\mathcal{F}$ be any nonempty family of open subsets of a metric space M. Then $\bigcup_{G \in \mathcal{F}} G$ is also an open subset of M. 3

OR

Q: 4 [A] Prove that Every open subset G of $\mathbb{R}$ can be written $G = \bigcup I_n$, where $I_1, I_2, I_3, \ldots$ are a finite number or a countable number of open intervals which are mutually disjoint. 5

[B] Prove that $[0, 1]$ and $[a, b]$ are homeomorphic. 3

Q: 5 [A] If the subset A of the metric space (M, ρ) is totally bounded, then prove that A is bounded. 5

[B] Prove that subset A of $\mathbb{R}$ is totally bounded *iff* A is bounded. 3

OR

Q: 5. Let (M, ρ) be a metric space. Then prove that a subset A of M is totally bounded *iff* every sequence of points of A contains a Cauchy subsequence. 8

Q: 6 [A] If A is a closed subset of the compact metric space (M, ρ), then prove that the metric space (A, ρ) is also compact. 5

[B] Prove that if f is a continuous function from a compact metric space M_1 into a metric space M_2 then the range $f(M_1)$ of f is also compact. 3

OR

Q: 6 [A] Let f be a one-one, continuous function on a compact metric space (M_1, ρ_1) onto (M_2, ρ_2). Then prove that f^{-1} is continuous and hence f is a homeomorphism of M_1 onto M_2. 5

[B] If a real valued function f is continuous on a compact metric space M, then prove that f attains the maximum and the minimum values at some points of M. 3