T.Y.B.Sc. : Semester - VI

US06CMTH23

Linear Algebra

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1. Inner Product

Inner product:
Let V be a real or complex vector space. The inner product on V is a binary operation . which associates each **ordered** pair u, v of vectors in V with a unique scalar $u.v$ satisfying following properties.

(i) $u.(v + w) = u.v + u.w$

(ii) $(\alpha u).v = \alpha(u.v)$

(iii) $u.v = \overline{v.u}$

(iv) $\overline{0}.u = 0 = u.\overline{0}$

2. Inner Product Space

Inner Product Space
A vector space together with an inner product defined on it, is called an Inner Product Space.

3. Euclidean Spaces and Unitary Spaces

Euclidean Spaces and Unitary Spaces
A finite dimensional **real inner product space** is called an **Euclidean Space** and a finite dimensional **complex inner product space** is called a **Unitary Space**.

4. **Let V be an inner product space u, v and w be any three vectors in V, and α a scalar. Then prove the following.**
(i) $(u+v).w = u.w + v.w$

(ii) $u.(\alpha v) = \overline{\alpha}(uv)$

(iii) $\overline{0}.u = 0 = u.\overline{0}$

Proof:
(i)

$$\begin{aligned}(u+v).w &= \overline{w.(u+v)} \\ &= \overline{w.u + w.v} \\ &= \overline{u.w} + \overline{v.w} \\ &= w.u + w.v \\ \therefore (u+v).w &= w.u + w.v\end{aligned}$$

(ii)

$$\begin{aligned}u.(\alpha v) &= \overline{(\alpha v).u} \\ &= \overline{\alpha(v.u)} \\ &= \overline{\alpha}(\overline{v.u}) \\ &= \overline{\alpha}(u.v) \\ \therefore u.(\alpha v) &= \overline{\alpha}(u.v)\end{aligned}$$

(ii)

$$\begin{aligned}\overline{0}.u &= (0v).u \\ &= 0(v.u) \\ &= 0 \\ \therefore \overline{0}.u &= 0\end{aligned}$$

$$\begin{aligned}u.\overline{0} &= u.(0v) \\ &= 0(u.v) \\ &= 0 \\ \therefore \overline{0}.u &= 0\end{aligned}$$

Hence,

$$\overline{0}.u = 0 = u.\overline{0}$$

5. **Norm**

Norm:
Let V be an inner product space. For $u \in V$ the norm of u, generally denoted by $normu$, is defined as

$$\|u\| = \sqrt{u.u}$$

> 6. **Let V be an inner product space. Then for arbitrary vectors u and v in V, and scalar α prove the following,**
>
> **(i)** $\|\alpha u\| = |\alpha|\|u\|$
>
> **(ii)** $\|u\| \geqslant 0$ **and** $\|u\| = 0$ **iff** $u = \overline{0}$
>
> **(iii)** $|u.v| \leqslant \|u\|\|v\|$
>
> **(vi)** $\|u+v\| = \|u\| + \|v\|$

Proof:
(i)

$$\begin{aligned}
\|\alpha u\| &= \sqrt{(\alpha u).(\alpha u)} \\
\therefore, \|\alpha u\|^2 &= (\alpha u).(\alpha u) \\
&= \alpha(u.(\alpha u)) \\
&= \alpha.\overline{\alpha}(u.u) \\
&= |\alpha|^2\|u\| \\
\therefore, \|\alpha u\| &= |\alpha|\|u\|
\end{aligned}$$

(ii)

$$\begin{aligned}
\|u\|^2 &= u.u \\
&\geqslant 0 \\
\therefore \|u\| \geqslant 0
\end{aligned}$$

Also,

$$\|u\| = 0 \Leftrightarrow \sqrt{u.u} = 0 \Leftrightarrow u.u = 0 \Leftrightarrow u = \overline{0}$$

(iii) If $u = \overline{0}$ then clearly, $|u.v| \leqslant \|u\|\|v\|$.

Now, if $u \neq \overline{0}$ then $u.u > 0$. Therefore $\|u\| > 0$

Define, a scalar $\alpha = \dfrac{v.u}{\|u\|^2}$.

Let $w = v - \alpha u$.

Now,

$$\begin{aligned}
0 &\leqslant w.w \\
&= (v-\alpha u).(v-\alpha u) \\
&= v.v - v.(\alpha u) - (\alpha u).v + (\alpha u).(\alpha u) \\
&= v.v - \overline{\alpha}(v.u) - \alpha(u.v) + (\alpha\overline{\alpha})u.u \\
&= v.v - \overline{\alpha}(v.u) - \alpha(u.v) + |\alpha|^2 u.u \\
&= \|v\|^2 - \frac{\overline{v.u}}{\|u\|^2}(v.u) - \frac{v.u}{\|u\|^2}(\overline{v.u}) + \frac{|u.v|^2}{\|u\|^4}\|u\|^2 \\
&= \|v\|^2 - 2\frac{|v.u|^2}{\|u\|^2} + \frac{|u.v|^2}{\|u\|^2} \\
&= \|v\|^2 - \frac{|v.u|^2}{\|u\|^2} \\
\therefore \frac{|v.u|^2}{\|u\|^2} &\leqslant \|v\|^2 \\
\therefore |u.v|^2 &\leqslant \|u\|^2\|v\|^2 \\
Hence, |u.v| &\leqslant \|u\|\|v\|
\end{aligned}$$

(iv)

$$\begin{aligned}
\|u+v\|^2 &= (u+v).(u+v) \\
&= u.u + u.v + v.u + v.v \\
&= \|u\|^2 + u.v + \overline{u.v} + \|v\|^2 \\
&= \|u\|^2 + 2Re(u.v) + \|v\|^2 \\
&\leqslant \|u\|^2 + 2|u.v| + \|v\|^2 \\
&= \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \\
&= (\|u\| + \|v\|)^2 \\
\therefore \|u+v\| &\leqslant \|u\| + \|v\|
\end{aligned}$$

7. Orthogonal Vectors

Orthogonal Vectors

Two vectors u, v of an inner product space V are said to be orthogonal to each other if

$$u.v = 0$$

8. Orthogonal set of vectors

Orthogonal set of non-zero Set of vectors

A subset of an inner product space is said to be an orthogonal set if for each pair of distinct vectors in the set is orthogonal.

> 9. **Prove that any orthogonal set of non-zero vectors in an inner product space is linearly independent (LI).**

Proof:
Let A be an orthogonal subset of an inner vector space and $B = \{u_1, u_2, \ldots, u_n\}$ is a finite subset of A.

Suppose, $\alpha_i,\ i = 1, 2, \ldots, n$ are scalars such that,

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = \overline{0}$$

Now, for any $u_i \in B$,

$$\begin{aligned}
\overline{0}.u_i &= 0 \\
(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n).u_i &= 0 \\
\alpha_1(u_1.u_i) + \alpha_2(u_2.u_i) + \cdots + \alpha_n(u_n.u_i) &= 0 \\
\alpha_i(u_i.u_i) &= 0 \ \ (\because u_i.u_j = 0,\ i \neq j) \\
\alpha_i &= 0 \text{ (As } u_i \neq \overline{0},\ u_i.u_i > 0)
\end{aligned}$$

Since, $\alpha_i = 0,\ \forall i = 1, 2, \ldots n$, the subset B of A is a linearly dependent set. Therefore, every finite subset of A is linearly independent. Hence A is linearly independent.

> 10. **Projection of a vector**

Projection of a vector
Let V be an inner product space and $u, v \in V$, where $v \neq \overline{0}$. The projection of u, along v is defined as the vector,

$$\frac{u.v}{\|v\|^2} v$$

> 11. **Prove that every finite dimensional inner product space V has an orthogonal basis.**

Proof:
Let $\{u_1, u_2, \ldots, u_n\}$ be a basis of an n-dimensional inner product space V. Using this basis we shall construct a basis $\{v_1, v_2, \ldots, v_n\}$ of V which is orthogonal.

Take $v_1 = u_1$. Now define v_2 by subtracting the projection of u_2 on v_1 from u_2 as given below,

$$v_2 = u_2 - \frac{u_2.v_1}{\|v_1\|^2} v_1$$

Here,

$$v_1.v_2 = v_1.u_2 - \frac{v_1.u_2}{\|v_1\|^2}(v_1.v_1) = v_1.u_2 - \frac{v_1.u_2}{\|v_1\|^2}\|v_1\|^2 = v_1.u_2 - v_1.u_2 = 0$$

Hence, v_1 and v_2 are orthogonal.

Next, define v_3 by subtracting projections of v_1 and v_2 on u_3 from u_3 as follows,

$$v_3 = u_3 - \frac{u_3.v_1}{\|v_1\|^2} v_1 - \frac{u_3.v_2}{\|v_2\|^2} v_2$$

As seen above here also we get,

$$v_1.v_3 = 0 \text{ and } v_2.v_3 = 0$$

Hence, v_1, v_2 and v_3 are orthogonal. Continuing similarly, in general we can construct v_k by

$$v_k = u_k - \sum_{i=1}^{k} \frac{u_k.v_i}{\|v_i\|^2} v_i$$

for $k = 1, 2, \ldots, n$. Such that

$$v_i.v_j = 0, \ i \neq j$$

Hence, $\{v_1, v_2, \ldots, v_n\}$ is an orthogonal set.

Also, none of v_k can be a zero vector, because for any $v_k = \overline{0}$ we can express u_k as a linear combination of $v_1, v_2, \ldots, v_k$, hence as a linear combination of $u_1, u_2, \ldots, u_k$. That is not possible as $\{u_1, u_2, \ldots, u_n\}$ is linearly independent.

Since, $\{v_1, v_2, \ldots, v_n\}$ is an orthogonal set of n non-zero vectors it is linearly independent. Hence it is an orthogonal basis for the n-dimensional vector space V.

12. Orthonormal set of vectors

Orthogonal set of non-zero Set of vectors

An orthogonal set V of non-zero vectors is said to be an **orthonormal** set if

$$\|u\| = 1 \ , \ \forall u \in V$$

13. Orthonormalise the set of linearly independent vectors $\{(1, 0, 1, 1), (-1, 0, -1, 1), (0, -1, 1, 1)\}$ of V_4.

Answer:

Suppose, $u_1 = (1,0,1,1), u_2 = (-1,0,-1,1)$ and $u_3 = (0,-1,1,1)$. First we shall find orthogonal vectors corresponding to u_1, u_2 and u_3.

Let $v_1 = u_1 = (1,0,1,1)$. Now construct v_2 as follows,

$$\begin{aligned}
v_2 &= u_2 - \frac{u_2.v_1}{\|v_1\|^2} v_1 \\
&= (-1,0,-1,1) - \frac{(-1,0,-1,1).(1,0,1,1)}{3}(1,0,1,1) \\
&= (-1,0,-1,1) + \frac{1}{3}(1,0,1,1) \\
\therefore, v_2 &= \left(-\frac{2}{3},\ 0,\ -\frac{2}{3},\ \frac{4}{3}\right)
\end{aligned}$$

Finally, construct v_3 as follows,

$$\begin{aligned}
v_3 &= u_3 - \frac{u_3.v_1}{\|v_1\|^2} v_1 - \frac{u_3.v_2}{\|v_2\|^2} v_2 \\
&= (0,-1,1,1) - \frac{(0,-1,1,1).(1,0,1,1)}{3}(1,0,1,1) - \frac{(0,-1,1,1).\left(-\frac{2}{3},\ 0,\ -\frac{2}{3},\ \frac{4}{3}\right)}{\frac{8}{3}}\left(-\frac{2}{3},\ 0,\ -\frac{2}{3},\ \frac{4}{3}\right) \\
&= (0,-1,1,1) - \frac{2}{3}(1,0,1,1) - \frac{1}{4}\left(-\frac{2}{3},\ 0,\ -\frac{2}{3},\ \frac{4}{3}\right) \\
\therefore, v_3 &= \left(-\frac{1}{2},\ -1,\ \frac{1}{2},\ 0\right)
\end{aligned}$$

Thus, we get the orthogonal set $\{v_1, v_2, v_3\}$.

Now, $\|v_1\| = \sqrt{3}$, $\|v_2\| = 2\sqrt{\frac{2}{3}}$ and $\|v_3\| = \sqrt{\frac{3}{2}}$. The orthonormal set can be obtained by dividing v_1, v_2 and v_3 by their respective norms as follows,

$$\left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\right\} = \left\{\left(\frac{1}{\sqrt{3}},\ 0,\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{6}},\ 0,\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)\left(-\frac{1}{\sqrt{3}},\ -\frac{2}{\sqrt{3}},\ -\frac{1}{\sqrt{3}},\ 0\right)\right\}$$

14. **Find an orthonormal basis of $P_3[-1,1]$ starting from the basis $\{1, x, x^2, x^3\}$ Use the inner product defined by**

$$f.g = \int_{-1}^{1} f(t)g(t)dt$$

Answer:
Suppose, $u_1 = 1, u_2 = x, u_3 = x^2, u_4 = x^3$. First we shall find orthogonal vectors corresponding

to u_1, u_2, u_3 and u_4.

Let $v_1 = u_1 = 1$. Now construct v_2 using $v_2 = u_2 - \frac{u_2.v_1}{\|v_1\|^2}v_1$
Here,

$$\begin{aligned}
u_2.v_1 &= \int_{-1}^{1} u_2(t)v_1(t).dt \\
&= \int_{-1}^{1} (t)\,(1)\,.dt \\
&= \int_{-1}^{1} t.dt \\
&= \left[\frac{1}{2}\,t^2\right]_{-1}^{1} \\
\therefore u_2.v_1 &= 0
\end{aligned}$$

Also

$$\begin{aligned}
||v_1||^2 &= v_1.v_1 \\
&= \int_{-1}^{1} v_1(t)v_1(t).dt \\
&= \int_{-1}^{1} (1)^2\,.dt \\
&= \int_{-1}^{1} 1.dt \\
&= [t]_{-1}^{1} \\
\therefore ||v_1|| &= 2
\end{aligned}$$

Therefore, we get, $\frac{u_2.v_1}{||v_1||^2}v_1 = \left(\frac{0}{2}\right) 1 = 0$ Let $v_2 = u_2 - \frac{u_2.v_1}{\|v_1\|^2}v_1$

Therefore, $v_2 = x - 0 = x$. Now, we calculate v_3 using $v_3 = u_2 - \frac{u_2.v_1}{\|v_1\|^2}v_1 - \frac{u_2.v_2}{\|v_2\|^2}v_2$ Here,

$$\begin{aligned}
u_3.v_1 &= \int\limits_{-1}^{1} u_3(t)v_1(t).dt \\
&= \int\limits_{-1}^{1} \left(t^2\right)(1).dt \\
&= \int\limits_{-1}^{1} t^2.dt \\
&= \left[\frac{1}{3}t^3\right]_{-1}^{1} \\
\therefore u_3.v_1 &= \frac{2}{3}
\end{aligned}$$

Also

$$\begin{aligned}
||v_1||^2 &= v_1.v_1 \\
&= \int\limits_{-1}^{1} v_1(t)v_1(t).dt \\
&= \int\limits_{-1}^{1} (1)^2.dt \\
&= \int\limits_{-1}^{1} 1.dt \\
&= [t]_{-1}^{1} \\
\therefore ||v_1|| &= 2
\end{aligned}$$

Therefore, we get, $\frac{u_3.v_1}{||v_1||^2}v_1 = \left(\frac{2/3}{2}\right)1 = \frac{1}{3}$ Also,

$$\begin{aligned}
u_3.v_2 &= \int\limits_{-1}^{1} u_3(t)v_2(t).dt \\
&= \int\limits_{-1}^{1} \left(t^2\right)(t).dt \\
&= \int\limits_{-1}^{1} t^3.dt \\
&= \left[\frac{1}{4}t^4\right]_{-1}^{1} \\
\therefore u_3.v_2 &= 0
\end{aligned}$$

Also

$$
\begin{aligned}
||v_2||^2 &= v_2.v_2 \\
&= \int\limits_{-1}^{1} v_2(t)v_2(t).dt \\
&= \int\limits_{-1}^{1} (t)^2 .dt \\
&= \int\limits_{-1}^{1} t^2.dt \\
&= \left[\frac{1}{3}\,t^3\right]_{-1}^{1} \\
\therefore ||v_2|| &= \frac{2}{3}
\end{aligned}
$$

Therefore, we get, $\frac{u_3.v_2}{||v_2||^2}v_2 = \left(\frac{0}{2/3}\right) x = 0$ Therefore, $v_3 = x^2 - \frac{1}{3} - 0 = x^2 - \frac{1}{3}$ Finally, we calculate v_4 using $v_4 = u_4 - \frac{u_4.v_1}{\|v_1\|^2}v_1 - \frac{u_4.v_2}{\|v_2\|^2}v_2 - \frac{u_4.v_3}{\|v_3\|^2}v_3$ Now,

$$
\begin{aligned}
u_4.v_1 &= \int\limits_{-1}^{1} u_4(t)v_1(t).dt \\
&= \int\limits_{-1}^{1} \left(t^3\right)(1) .dt \\
&= \int\limits_{-1}^{1} t^3.dt \\
&= \left[\frac{1}{4}\,t^4\right]_{-1}^{1} \\
\therefore u_4.v_1 &= 0
\end{aligned}
$$

Also

$$
\begin{aligned}
||v_1||^2 &= v_1.v_1 \\
&= \int\limits_{-1}^{1} v_1(t)v_1(t).dt \\
&= \int\limits_{-1}^{1} (1)^2 .dt \\
&= \int\limits_{-1}^{1} 1.dt \\
&= [t]_{-1}^{1} \\
\therefore ||v_1||^2 &= 2
\end{aligned}
$$

Therefore, we get, $\dfrac{u_4.v_1}{||v_1||^2}v_1 = \left(\dfrac{0}{2}\right) 1 = 0$ Also,

$$
\begin{aligned}
u_4.v_2 &= \int\limits_{-1}^{1} u_4(t)v_2(t).dt \\
&= \int\limits_{-1}^{1} \left(t^3\right)(t).dt \\
&= \int\limits_{-1}^{1} t^4.dt \\
&= \left[\frac{1}{5}\,t^5\right]_{-1}^{1} \\
\therefore u_4.v_2 &= \frac{2}{5}
\end{aligned}
$$

Also

$$
\begin{aligned}
||v_2||^2 &= v_2.v_2 \\
&= \int\limits_{-1}^{1} v_2(t)v_2(t).dt \\
&= \int\limits_{-1}^{1} (t)^2\,.dt \\
&= \int\limits_{-1}^{1} t^2.dt \\
&= \left[\frac{1}{3}\,t^3\right]_{-1}^{1} \\
\therefore\ ||v_2||^2 &= \frac{2}{3}
\end{aligned}
$$

Therefore, we get, $\dfrac{u_4.v_2}{||v_2||^2}v_2 = \left(\dfrac{2/5}{2/3}\right)x = \frac{3}{5}\,x$ Also,

$$
\begin{aligned}
u_4.v_3 &= \int\limits_{-1}^{1} u_4(t)v_3(t).dt \\
&= \int\limits_{-1}^{1} \left(t^3\right)\left(t^2 - \frac{1}{3}\right).dt \\
&= \int\limits_{-1}^{1} \frac{1}{3}\left(3\,t^2 - 1\right)t^3.dt \\
&= \left[\frac{1}{6}\,t^6 - \frac{1}{12}\,t^4\right]_{-1}^{1} \\
\therefore\ u_4.v_3 &= 0
\end{aligned}
$$

Also

$$\begin{aligned}
||v_3||^2 &= v_3.v_3 \\
&= \int_{-1}^{1} v_3(t)v_3(t).dt \\
&= \int_{-1}^{1} \left(t^2 - \frac{1}{3}\right)^2 .dt \\
&= \int_{-1}^{1} \frac{1}{9}\left(3\,t^2 - 1\right)^2 .dt \\
&= \left[\frac{1}{5}\,t^5 - \frac{2}{9}\,t^3 + \frac{1}{9}\,t\right]_{-1}^{1} \\
\therefore ||v_3||^2 &= \frac{8}{45}
\end{aligned}$$

Therefore, we get, $\frac{u_4.v_3}{||v_3||^2}v_3 = \left(\frac{0}{8/45}\right)x^2 = 0$

Now, $v_4 = u_4 - \frac{u_4.v_1}{\|v_1\|^2}v_1 - \frac{u_4.v_2}{\|v_2\|^2}v_2 - \frac{u_4.v_3}{\|v_3\|^2}v_3$

Therefore, $v_4 = x^3 - 0 - \frac{3}{5}x - 0 = x^3 - \frac{3x}{5}$

Thus, we obtain orthogonal set $\left\{1, x, x^2 - \frac{1}{3}, x^3 - \frac{3x}{5}\right\}$. Now, to othonormalize the vectors we shall divide each vector with its norm. We have Let us calculate $\|v_4\|$.

$$\begin{aligned}
||v_4||^2 &= v_4.v_4 \\
&= \int_{-1}^{1} v_4(t)v_4(t).dt \\
&= \int_{-1}^{1} \left(t^3 - \frac{3}{5}\,t\right)^2 .dt \\
&= \int_{-1}^{1} \frac{1}{25}\left(5\,t^3 - 3\,t\right)^2 .dt \\
&= \left[\frac{1}{7}\,t^7 - \frac{6}{25}\,t^5 + \frac{3}{25}\,t^3\right]_{-1}^{1} \\
\therefore ||v_4||^2 &= \frac{8}{175}
\end{aligned}$$

Dividing v_1, v_2, v_3 and v_4 with their respective norms, we get the orthonormal set,

$$\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right), \frac{5\sqrt{7}}{2\sqrt{2}}\left(x^3 - \frac{3x}{5}\right)\right\}$$

15. **A real (complex) square matrix is orthogonal (unitary) iff the rows of the matrix form an orthonormal set of vectors or iff the columns of the matrix form an orthonormal set of vectors.**