T.Y.B.Sc. : Semester - V (CBCS)

US05CMTH24

Metric Spaces and Topological Spaces

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Study Material Prepared by :
Mr. Rajesh P. Solanki
Department of Mathematics and Statistics
V.P. and R.P.T.P. Science College, Vallabh Vidyanagar

US05CMTH24- UNIT : II

1. Topology

Topology

Let X be a non-empty set. A collection $\mathcal{T}$ of subsets of X is said to be a topology for X if the following properties are satisfied.
(i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
(ii) If $G = \{G_\alpha \ / \ \alpha \in \Lambda\}$ is an arbitrary collection of members of $\mathcal{T}$, then

$$\bigcup_{\alpha \in \Lambda} G_\alpha \in \mathcal{T}$$

(iii) If $G = \{G_i \ / \ i = 1, 2, \ldots, n\}$ is a finite collection of members of $\mathcal{T}$ then

$$\bigcap_{i=1}^{n} G_i \in \mathcal{T}$$

Every member of $\mathcal{T}$ is called a $\mathcal{T}$-Open set and the set X with the topology $\mathcal{T}$ defined on it is called a topological space which is denoted by $(X, \mathcal{T})$.

2. Indiscrete Topology

Indiscrete Topology

For a non-empty set X the topology $\{\emptyset, X\}$ is called the indiscrete topology and usually it is denoted by $\mathcal{I}$

3. Show that indiscrete topology satisfies all the conditions for becoming a topological space

Proof

Let X be a non-empty set. Then the indiscrete topology for X is $\mathcal{I} = \{\emptyset, X\}$.
Now we show that $\mathcal{I}$ possesses all the properties to become a topology.

By the definition of $\mathcal{I}$ we have

$$\emptyset \in \mathcal{I} \text{ and } X \in \mathcal{I} \quad \text{- - - - (i)}$$

Now the union of an arbitrary collection of members of $\mathcal{I}$ is either $\emptyset$ or X.
Therefore $\mathcal{I}$ contains union of arbitrary collections of members of $\mathcal{I}$.

Finally, intersection of any collection of members of $\mathcal{I}$ is either $\emptyset$ or X.

Therefore $\mathcal{I}$ contains intersection of finite collections of members of $\mathcal{I}$.

Thus, $\mathcal{I}$ possesses all the properties for becoming a topology.

4.	**Discrete Topology**

Discrete Topology

For a non-empty set X the collection of all the subsets of X is called the discrete topology and usually it is denoted by $\mathcal{D}$

5.	**Show that discrete topology satisfies all the conditions for becoming a topological space**

Proof

Let X be a non-empty set. Then the discrete topology $\mathcal{D}$ for X is the collection of all the subsets of X.
Now we show that $\mathcal{D}$ possesses all the properties to become a topology.

By the definition of $\mathcal{D}$ we have

$$\emptyset \in \mathcal{D} \text{ and } X \in \mathcal{D} \quad ---- \text{(i)}$$

Next consider an arbitrary collection $\{G_\alpha \ / \ \alpha \in \Lambda\}$ of members of $\mathcal{D}$.
As union of subsets of a set is also subset of that set, we have,

$$\bigcup_{\alpha\in\Lambda} G_\alpha \subset X$$

Hence,

$$\bigcup_{\alpha\in\Lambda} G_\alpha \in \mathcal{D} \quad ---- \text{(ii)}$$

Finally, consider a finite collection $G = \{G_i \ / \ i = 1, 2, \ldots, n\}$ of members of $\mathcal{D}$.
As intersection of subsets of a set is also subset of that set, we have,

$$\bigcap_{i=1}^{n} G_i \subset X$$

Hence,

$$\bigcap_{i=1}^{n} G_i \in \mathcal{D} \quad ---- \text{(iii)}$$

From (i),(ii) and (iii) it follows that $\mathcal{D}$ possesses all the properties for becoming a topology.

6.	**What are trivial topologies on a non-empty set?**

Trivial Topologies:

The following topologies are defined for every non-empty set and collectively they are called Trivial Topologies.

Discrete Topology For a non-empty set X the collection of all the subsets of X is called the discrete topology and usually it is denoted by $\mathcal{D}$ Indiscrete Topology For a non-empty set X the topology $\{\emptyset, X\}$ is called the indiscrete topology and usually it is denoted by $\mathcal{I}$

7. $\mathcal{U}$-Open Set

$\mathcal{U}$-Open Set:
A set $G \subset R$ is said to be $\mathcal{U}$-open set
(I) if $G = \emptyset$ or
(ii) if $G \neq \emptyset$ then for each $p \in G$ there is an open interval I such that

$$p \in I \subset G$$

.

8. Usual Topology of R

Usual Topology of R:
The family $\mathcal{U}$ of all the subsets G of R as described below is called the Usual Topology for R.
(I) $G = \emptyset$ or
(ii) if $G \neq \emptyset$ then for each $p \in G$ there is an open interval I such that

$$p \in I \subset G$$

.

9. Show that usual topology of R possesses all the properties for becoming a topology for $\mathbb{R}$

Answer:
The usual topology $\mathcal{U}$ for R is defined as a family of subsets G of R as described below,
(i) $G = \emptyset$ or
(ii) if $G \neq \emptyset$ then for each $p \in G$ there is an open interval I such that

$$p \in I \subset G$$

Let us show that $\mathcal{U}$ possesses all the properties to be a topology for R.

(1) By the definition of $\mathcal{U}$ we have $\emptyset \in \mathcal{U}$

Also for any $p \in R$ and any $r > 0$ we have,

$$p \in (p - r, p + r) \subset R$$

Therefore, $R \in \mathcal{U}$.

Thus, we have, $\emptyset \in \mathcal{U}$ and $R \in \mathcal{U}$

(2) Next, consider an arbitrary family $\{G_\alpha \ / \ \alpha \in \Lambda\}$ of memmbers of $\mathcal{U}$.

If $p \in \bigcup\limits_{\alpha\in\Lambda} G_\alpha$ then for some $\alpha_p \in \Lambda$ we have

$$p \in G_{\alpha_p}$$

As G_{α_p} is a non-empty $\mathcal{U}$-open subset of R there is some open interval I such that

$$p \in I \subset G_{\alpha_p}$$

This implies that,

$$p \in I \subset \bigcup_{\alpha\in\Lambda} G_\alpha$$

Hence,

$$\bigcup_{\alpha\in\Lambda} G_\alpha \in \mathcal{U}$$

(3) Finally consider a finite family $\{G_1 G_2, \dots, G_n\}$ of members of $\mathcal{U}$.

If $p \in \bigcap\limits_{i=1}^{n} G_i$ then $p \in G_i, \ \forall i = 1, 2, \dots \ n$.

As each G_i is a non-empty member of $\mathcal{U}$, there must be some I_i such that,

$$p \in I_i \subset G_i, \ \forall i = 1, 2, \dots \ n$$

If we take $I = \bigcap\limits_{i=1}^{n} I_i$ then I is an open interval containg p such that $I \subset I_i$.
Therefore,

$$p \in I \subset G_i, \ \forall i = 1, 2, \ \dots \ n$$

Therefore,

$$p \in I \subset \bigcap_{i=1}^{n} G_i$$

Hence,

$$\bigcap_{i=1}^{n} G_i \in \mathcal{U}$$

From (1),(2) and (3) it follows that, $\mathcal{U}$ is a topology for R.

10. **Let $\mathcal{G}$ be a family of subsets of R as described below**
(i) $\emptyset \in \mathcal{G}$
(ii) If $G \neq \emptyset$ then $G \in \mathcal{G}$ if for each $p \in G$ there is a set $H = \{x \in R / a \leqslant x \ < b\}$ for some $a < b$ such that $p \in H \subset G$.
Prove that $\mathcal{G}$ is an unusual nontrivial topology of $\mathbb{R}$

Proof:

(1) By the definition of $\mathcal{G}$ we have $\emptyset \in \mathcal{G}$

Also, if $p \in R$ then we have $H = \{x \in R \ / \ p \leqslant x < p+1\}$ such that

$$p \in H \subset R$$

Therefore, $R \in \mathcal{G}$

(2) Next, consider an arbitrary family $\{G_\alpha \ / \ \alpha \in \Lambda\}$ of memmbers of $\mathcal{G}$.

If $p \in \bigcup\limits_{\alpha \in \Lambda} G_\alpha$ then for some $\alpha_p \in \Lambda$ we have

$$p \in G_{\alpha_p}$$

As G_{α_p} is a non-empty member of $\mathcal{G}$, there is some subset $H = \{x \in R \ /a \leqslant x < b\}$ such that

$$p \in H \subset G_{\alpha_p}$$

This implies that,

$$p \in H \subset \bigcup_{\alpha \in \Lambda} G_\alpha$$

Hence,

$$\bigcup_{\alpha \in \Lambda} G_\alpha \in \mathcal{G}$$

(3) Finally consider a finite family $\{G_1 G_2, \ldots, G_n\}$ of members of $\mathcal{G}$.

If $p \in \bigcap\limits_{i=1}^{n} G_i$ then $p \in G_i, \ \forall i = 1, 2, \ldots \ n$.

As each G_i is a non-empty member of $\mathcal{G}$, there must be some $H = \{x \in R \ /a_i \leqslant x < b_i\}$ such that,

$$p \in H_i \subset G_i, \ \forall i = 1, 2, \ldots \ n$$

Let,

$$a = max\{a_1, a_2, \ldots a_n\} \quad \text{and} \quad b = min\{b_1, b_2, \ldots b_n\}$$

clearly

$$a_i \leqslant a \quad \text{and} \quad b \leqslant b_i$$

Therefore, if we take $H = \{x \in R \ /a \leqslant x < b\}$ then we have,

$$p \in H \subset G_i, \ \forall i = 1, 2, \ \ldots \ n$$

Therefore,

$$p \in H \subset \bigcap_{i=1}^{n} G_i$$

Hence,

$$\bigcap_{i=1}^{n} G_i \in \mathcal{G}$$

From (1),(2) and (3) it follows that, $\mathcal{G}$ is a topology for R.

11. Let J be the set of all integers and $\mathcal{J}$ be a collection of subsets G of J where $G \in \mathcal{J}$ whenever $G = \emptyset$ or $G \neq \emptyset$ and $p, p\pm 2, p\pm 4, ..., p\pm 2n, ...$ belong to G whenever $p \in G$. Prove that $\mathcal{J}$ is a topology for J

Proof:

Here J is the set of all integers.
Therefore for every $p \in J$, we have

$$p, p \pm 2, p \pm 4, ..., p \pm 2n, \cdots \in J$$

Therefore,

$$J \in \mathcal{J} \quad \text{- - - - (i)}$$

Next, consider an arbitrary collection $\mathcal{G} = \{G_\alpha \;/\; \alpha \in \Lambda\}$ of members of $\mathcal{J}$.
If $p \in \bigcup_{\alpha\in\Lambda} G_\alpha$ then for some $\alpha_p \in \Lambda$, we have $p \in G_{\alpha_p}$
Since, $G_{\alpha_p} \in \mathcal{J}$, it follows that

$$p, p \pm 2, p \pm 4, \ldots, p \pm 2n, \cdots \in G_{\alpha_p}$$

Therefore,

$$p, p \pm 2, p \pm 4, ..., p \pm 2n, \cdots \in \bigcup_{\alpha\in\Lambda} G_\alpha$$

Hence,

$$\bigcup_{\alpha\in\Lambda} G_\alpha \in \mathcal{J} \quad \text{- - - - (ii)}$$

Finally, consider a finite collection $G = \{G_i \;/\; i = 1, 2, \ldots, n\}$ of members of $\mathcal{J}$.
If $p \in \bigcap_{i=1}^{n} G_i$ then $p \in G_i$, for every $i = 1, 2, \ldots n$
Since, $G_i \in \mathcal{J}$, $\forall i$, it follows that

$$p, p \pm 2, p \pm 4, ..., p \pm 2n, \cdots \in G_i, \quad \forall i$$

Therefore,

$$p, p \pm 2, p \pm 4, ..., p \pm 2n, \cdots \in \bigcap_{i=1}^{n} G_i$$

Hence,

$$\bigcap_{i=1}^{n} G_i \in \mathcal{J} \quad \text{- - - - (iii)}$$

From (i),(ii) and (iii) it follows that $\mathcal{J}$ a topology for J, which is non-trivial topology also.

12. Coarser Topology and Finer Topology.

Coarser Topology and Finer Topologiey

If T_1 and T_2 are two topologies for a non-empty set X such that

$$T_1 \subset T_2$$

then topology T_1 is called coarser than the topology T_2 and T_2 is called finer than T_1.

Collectively T_1 and T_2 are called comparable topologies.

13. Non-comparable topologies

Non-comparable topologies
If T_1 and T_2 are two topologies for a non-empty set X such that

$$T_1 \not\subset T_2 \text{ and } T_2 \not\subset T_1$$

then the two topologies are said to be non-comparable topologies.

14. Consider the topology $\mathcal{G}$ on R where $G \subset R$ is $\mathcal{G}$-open if $G = \emptyset$ or $G \neq \emptyset$ and for each $p \in G$ there is a set $H = \{x \in R/a \leqslant x \ < b\}$ for some $a < b$ such that $p \in H \subset G$. Prove that $\mathcal{G}$ is finer than usual topology of R.

Proof
To show that $\mathcal{G}$ is finer than usual topology $\mathcal{U}$ we shall prove that $\mathcal{U} \subset \mathcal{G}$
Let $G \in \mathcal{U}$.

Therefore, G is $\mathcal{U}$-open in R.

In case $G = \emptyset$ then $G \in \mathcal{G}$

Now, if $G \neq \emptyset$ then consider any $p \in G$

As G is $\mathcal{U}$-open, there is some open interval I such that

$$p \in I \subset G$$

.
If $I = (a, b)$ then we have $p \in (a, b)$.

Hence,

$$p \in [p, b)$$

If we take $H = \{x \in R/p \leq x < b\}$ then, $p \in H \subset G$

Therefore,

$$G \in \mathcal{G}$$

Therefore,

$$\mathcal{U} \subset \mathcal{G}$$

Hence, $\mathcal{G}$ is finer than $\mathcal{U}$

15. Closed Set

Closed Set:
Let (X, T) be a topological space. A subset A of X is called a T-closed set if there is a T-open subset G of X such that

$$A = X - G$$

16. If $(X, \mathcal{T})$ is a topological space and $\{F_\alpha / \alpha \in \Lambda\}$ is any collection of $\mathcal{T}$-closed subsets of X then prove that $\bigcap\{F_\alpha / \alpha \in \Lambda\}$ is a $\mathcal{T}$-closed set

Proof:
Let $(X, \mathcal{T})$ be a topological space and $\{F_\alpha \ / \ \alpha \in \Lambda\}$ be an arbitrary collection of $\mathcal{T}$-closed subsets of X
To prove that $\bigcap\limits_{\alpha \in \Lambda} F_\alpha$ is a $\mathcal{T}$-closed set we shall show that $X - \bigcap\limits_{\alpha \in \Lambda} F_\alpha$ is a $\mathcal{T}$-open set.

First we note that for each $\mathcal{T}$-closed set F_α in the collection its complement $X - F_\alpha$ is $\mathcal{T}$-open.

Now by DeMorgan's Law,

$$X - \bigcap_{\alpha \in \Lambda} F_\alpha = \bigcup_{\alpha \in \Lambda} (X - F_\alpha)$$

As the RHS is an arbitrary union of $\mathcal{T}$-open sets, it is a $\mathcal{T}$-open set.

Therefore $X - \bigcap\limits_{\alpha \in \Lambda} F_\alpha$ is $\mathcal{T}$-open.

Hence, $\bigcap\limits_{\alpha \in \Lambda} F_\alpha$ is $\mathcal{T}$-closed.

17. If $(X, \mathcal{T})$ is a topological space and $F_1, F_2, ..., F_n$ are $\mathcal{T}$-closed subsets of X then prove that $\bigcup\{F_i / i \in J_n\}$ is a $\mathcal{T}$-closed set

Proof:
Let $(X, \mathcal{T})$ be a topological space and $F_1, F_2, ..., F_n$ be $\mathcal{T}$-closed subsets of X.
To prove that $\bigcup\limits_{i=1}^{n} F_i$ is a $\mathcal{T}$-closed set we shall show that $X - \bigcup\limits_{i=1}^{n} F_i$ is a $\mathcal{T}$-open set.

Now by DeMorgan's Law,

$$X - \bigcup_{i=1}^{n} F_i = \bigcap_{i=1}^{n} (X - F_i)$$

Also each $X - F_i$ is $\mathcal{T}$-open as each F_i is $\mathcal{T}$-closed.
Therefore $X - \bigcup\limits_{i=1}^{n} F_i$ is $\mathcal{T}$-open as it is a finite intersection of $\mathcal{T}$-open sets.
Hence, $\bigcup\limits_{i=1}^{n} F_i$ is $\mathcal{T}$-closed.

18. Show in two ways that if $a \in R$ then $\{a\}$ is a closed set in usual topology of R.

Answer:
For any $a \in R$, consider the singletone set $\{a\}$.

We know that for any positive integer i the intervals $(a-i, a)$ and $(a, a+i)$ both are $\mathcal{U}$-open. Now, we can express the complement of $\{a\}$ as follows

$$R - \{a\} = \left(\bigcup_{i \in J^+} (a-i, a)\right) \bigcup \left(\bigcup_{i \in J^+} (a, a+i)\right)$$

As, we can express $R - \{a\}$ as a union of $\mathcal{U}$-open sets, it is $\mathcal{U}$-open.

Therefore, $\{a\}$ is a $\mathcal{U}$-closed set.

Also we can express $\{a\}$ as follows

$$\{a\} = [a-1, a] \cap [a, a+1]$$

Therefore $\{a\}$ is an intersetion of two $\mathcal{U}$ subsets of R. Hence $\{a\}$ is $\mathcal{U}$-closed.

19. Are closed intervals of R, $\mathcal{U}$-closed? where $\mathcal{U}$ is the usual topology for R

Answer:
Yes. closed intervals of R is a $\mathcal{U}$-closed set.

For any $a < b \in R$, consider a closed interval $[a, b]$.

We know that for any positive integer i the intervals $(a-i, a)$ and $(b, b+i)$ both are $\mathcal{U}$-open. Now, we can express the complement of $[a, b]$ as follows

$$R - [a, b] = \left(\bigcup_{i \in J^+} (a-i, a)\right) \bigcup \left(\bigcup_{i \in J^+} (b, b+i)\right)$$

As, we can express $R - [a, b]$ as a union of $\mathcal{U}$-open sets, it is $\mathcal{U}$-open.

Therefore, $[a, b]$ is $\mathcal{U}$-closed.

20. For the usual topology $\mathcal{U}$, show that half-open intervals of $\mathbb{R}$ are neither $\mathcal{U}$-open nor $\mathcal{U}$-closed

Answer:
For any $a < b$ consider a half-closed and half-open interval $[a, b)$

First we show that $[a, b)$ cannot be open.

If I is an open interval containing a then I contains infinitely many points less than a.

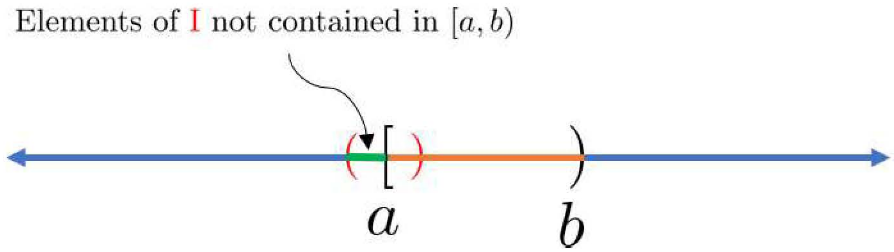

Figure 1: A Half Closed - Half Open Interval

(See figure ??)

Therefore, for any open interval I containing a we have

$$I \not\subset [a, b)$$

Hence, $[a, b)$ is not a $\mathcal{U}$ open set.

Also we have the complement of $[a, b)$

$$R - [a, b) = (-\infty, a) \cup [b, \infty)$$

Here also if I is any open interval such that $b \in I$ then I contains infinitely many points less

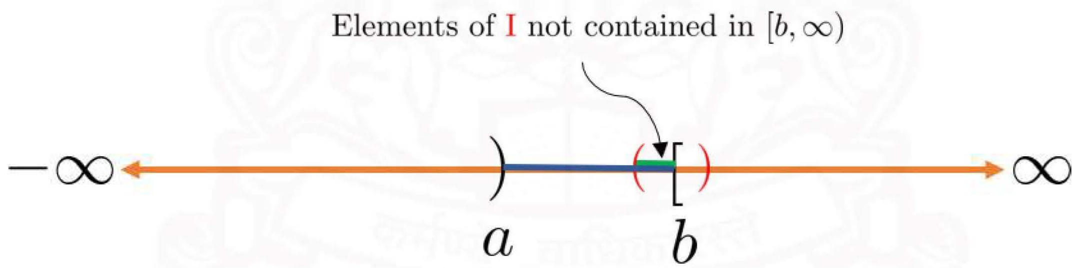

Figure 2: Complement of a Half Closed - Half Open Interval

than b not contained in $[b, \infty)$. (See figure ??)
Therefore,

$$I \not\subset [b, \infty)$$

Hence, for $b \in (-\infty, a) \cup [b, \infty)$, if I is any open interval containing b then

$$I \not\subset (-\infty, a) \cup [b, \infty)$$

Hence, $(-\infty, a) \cup [b, \infty)$ is not an open set.

Therefore, $R - [a, b)$ cannot be a $\mathcal{U}$-open set.

Hence, $[a, b)$ is not a $\mathcal{U}$-closed set.

Thus, $[a, b)$ is neither $\mathcal{U}$-open nor $\mathcal{U}$-closed in R.

21. Show that any finite set of real numbers is closed in the usual topology of $\mathbb{R}$

Answer:
First we show that every singleton subset of R is $\mathcal{U}$-closed.

Conseder any $a \in R$ and corrsponding singletone set $\{a\}$.

We know that for any positive integer i the intervals $(a - i, a)$ and $(a, a + i)$ both are $\mathcal{U}$-open. Now, we can express the complement of $\{a\}$ as follows

$$R - \{a\} = \left(\bigcup_{i \in J^+} (a - i, a) \right) \bigcup \left(\bigcup_{i \in J^+} (a, a + i) \right)$$

As, we can express $R - \{a\}$ as a union of $\mathcal{U}$-open sets, it is $\mathcal{U}$-open.

Therefore, $\{a\}$ is a $\mathcal{U}$-closed set.

Now, let A be any finite subset of R.

Suppose,

$$A = \{a_1, a_2, \ldots, a_n\}$$

Then we can express A as follows.

$$A = \bigcup_{i=1}^{n} \{a_i\}$$

As each singleton a_i is a $\mathcal{U}$-closed subset of R the set A has been represented as a finite union of $\mathcal{U}$-closed sets.

Hence A is $\mathcal{U}$-closed subset of R.

22. Neighbourhood of a point

Neighbourhood of a point :
Let (X, T) be a topological space and $x \in X$. A subset N of X is called a neighbourhood of x if there is a T-open set A such that

$$x \in A \subset N$$

23. Let $(X, \mathcal{T})$ be a topological space and let A be a subset of X. Prove that A is $\mathcal{T}$-open set iff A contains a $\mathcal{T}$-neighbourhood of each of its points

Proof:
Suppose a subset A of X contains a $\mathcal{T}$-neighbourhood of each of its points.
Therefore for each $p \in A$ there is a $\mathcal{T}$-neighbourhood N_p of p such that

$$N_p \subset A$$

As N_p is a $\mathcal{T}$-neighbourhood of p there exists some $\mathcal{T}$-open set G_p such that

$$p \in G_p \subset N_p$$

Let

$$G = \bigcup_{p \in A} G_p$$

Here G is a $\mathcal{T}$-open set as it is a union of $\mathcal{T}$-open sets.

Now we show that $A = G$

For any $x \in A$, as discussed above, there is some T-open set G_x such that $x \in G_x$
Therefore,

$$x \in \bigcup_{p \in A} G_p = G$$

Hence,

$$A \subset G$$

Also, if $x \in G$ then $x \in G_p$ for some $p \in A$.
As $G_p \subset N_p \subset A$, we have $x \in A$ Hence,

$$G \subset A$$

Thus,

$$A = G$$

Since G is $\mathcal{T}$-open, we conclude that A is $\mathcal{T}$-open.

Conversely suppose A is T-open. Then, A is a T neighbourhood of each of its points

Since, $A \subset A$, A contains a T-neighbourhood of each of its points.

24. If $X = \{a, b, c\}$ then find three topologies $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ for X such that $\mathcal{T}_1 \subsetneqq \mathcal{T}_2 \subsetneqq \mathcal{T}_3$

Answer:
Possible topologies satisfying the conditions are
$T_1 = \{\emptyset, X\}$
$T_2 = \{\emptyset, X, \{a\}\}$
$T_3 = \{\emptyset, X, \{a\}, \{a, b\}\}$

25. Find three mutually non-comparable topologies of $X = \{a, b, c\}$

Answer:
Following are three mutually non-comparable topologies of $X = \{a, b, c\}$

$$T_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$$
$$T_2 = \{\emptyset, X, \{b\}, \{a, b\}\}$$
$$T_3 = \{\emptyset, X, \{c\}, \{a, c\}\}$$

26. Door space

A topological space (X, T) is known as a Door Space if each subset of X is either a T-open set or a T-closed set.

27. Give an example of a Door Space

An example of a Door Space
Consider a set $X = \{a, b, c\}$ and a topology $\mathcal{T}$ for it given by

$$\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$$

All the subsets of X other than the members of topology $\mathcal{T}$ are given in the following collection.

$$\{\{c\}, \{b, c\}, \{a, c\}\}$$

We observe that,

$$\{c\} = X - \{a, b\}, \quad \text{hence it is a } \mathcal{T} - \text{closed subset of } X$$
$$\{b, c\} = X - \{a\}, \quad \text{hence it is a } \mathcal{T} - \text{closed subset of } X$$
$$\{a, c\} = X - \{b\}, \quad \text{hence it is a } \mathcal{T} - \text{closed subset of } X$$

Thus, every subset of X is eiter a $\mathcal{T}$-open set or a $\mathcal{T}$-closed set.

Hence, (X, T) is a Door Space.

28. Which of the following subsets of $\mathbb{R}$ are $\mathcal{U}$-neighbourhood of 2?

(a) $(1, 3)$ **(b)** $[1, 3)$ **(c)** $[2, 3)$ **(d)** $[1, 3]$ **(e)** $[1, 3] - 2\frac{1}{8}$
(f) $(2, 3)$ **(g)** $(1, 3]$ **(h)** $[2, 3]$ **(i)** $\mathbb{R}$

Solution
(a) As $2 \in (1.5, 2.5) \subset (1, 3)$, the set $(1, 3)$ is a $\mathcal{U}$-neighbourhood of 2.

(b) As $2 \in (1.5, 2.5) \subset [1, 3)$, the set $[1, 3)$ is a $\mathcal{U}$-neighbourhood of 2.

(c) Any open interval I containing 2 has infinitely many real numbers on its left-hand side. Therefore

$$I \not\subset [2,3)$$

Hence, $[2,3)$ cannot be $\mathcal{U}$-neighbourhood of 2.

(d) As $2 \in (1.5, 2.5) \subset [1,3]$, the set $[1,3]$ is a $\mathcal{U}$-neighbourhood of 2.

(e) As $2 \in (2 + \frac{1}{16}, 2 - \frac{1}{16}) \subset [1,3] - 2\frac{1}{8}$, the set $[1,3] - 2\frac{1}{8}$ is a $\mathcal{U}$-neighbourhood of 2.

(f) As 2 is not contained in $(2,3)$ it cannot be $\mathcal{U}$-neighbourhood of 2.

(g) As $2 \in (1.5, 2.5) \subset (1,3]$, the set $(1,3]$ is a $\mathcal{U}$-neighbourhood of 2.

(h) Any open interval I containing 2 has infinitely many real numbers on its left-hand side. Therefore

$$I \not\subset [2,3]$$

Hence, $[2,3]$ cannot be $\mathcal{U}$-neighbourhood of 2.