T.Y.B.Sc. : Semester - V (CBCS)

US05CMTH24

Metric Spaces and Topological Spaces

[Syllabus effective from June , 2020]

Study Material Prepared by :
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Unit:1
Countability of Sets, Metric spaces , Limit in metric spaces , continuous functions on a Metric space, Open and Closed Sets.

Unit:2
Topological Spaces : Definition and examples , Open and close sets in topological spaces , Usual topology on R , Comparison of topologies , Neighbourhood.

Unit:3
Cluster points , Closure and interior points of a set , Definition and examples of a door space and dense set , Continuity in a topological space and homeomorphism.

Unit:4
Definition and examples of connected and disconnected spaces , Connectedness in R , Relative topology , Connected subspaces , Open cover, Compact space.

Recommended Textbooks :

1. Methods of Real Analysis
   Author : Richard R. Goldberg
   Edition : Revised Ed., 1970
   Publisher : Oxford & IBH Publishing Co. Pvt. Ltd, New Delhi

2. Introduction to Topology
   Author : M.J. Mansfield
   Edition : Revised Ed., 1970
   Publisher : CBS Publisher & Distributers, Delhi

Recommended Reference Books :

1. Mathematical Analysis
   Author : S.C.Malik and Savita Arora
   Edition : 2nd
   Publisher : New Age International Pvt. Ltd., New Delhi

US05CMTH24- UNIT : I

1. Equivalent Sets

Equivalent Sets
If there exists a one-to-one correspondance between two sets then they are called equivalent to each other.

2. Infinite Set

Infinite Set
A set is called an infinite set if for any positive integer n there is a subset of the set containing exactly n elements.

3. Countable Set (or Denumerable Set)

Countable Set (or Denumerable Set)
A set is called a Countable or Denumerable set if it is equivalent to the set of postive integers.

4. Uncountable Set.

Uncountable Set
An infinite set which is not countable is called an uncountable set.

5. Prove that the set of all integers is conuntable.

Proof:
The set of all ittegers is

$$Z = \{\dots, -4, -3, -2, -1, -0, 1, 2, 3, 4, \dots\}$$

Define a function $f : Z \to I$ by,

$$f(n) = \begin{cases} \dfrac{n-1}{2}, & \text{for } \; n = 1, 3, 5, 7, \dots \\ -\dfrac{n}{2}, & \text{for } \; n = 2, 4, 6, 8, \dots \end{cases}$$

Here f is one-one and onto I, the set of positive integers. Therefore f is a one-to-one correspondance between Z and I. Hence, Z is countable.

6. If $A_1, A_2, \ldots$ are countable sets then prove that $\bigcup_{i=1}^{\infty} A_i$ is also countable.

Proof:
Each of $A_1, A_2, \ldots$ is countable a set. So we can arrange them in an order corresponding to the order of the positive integers. Suppose,

$$\begin{aligned} A_1 &= \{a_1^1, a_2^1, a_3^1, \ldots\} \\ A_2 &= \{a_1^2, a_2^2, a_3^2, \ldots\} \\ &\vdots \\ A_n &= \{a_1^n, a_2^n, a_3^n, \ldots\} \\ &\vdots \end{aligned}$$

Now, for any element a_i^j of a set A_j define

$$\text{Height of } a_i^j = i + j$$

Next, we arrange the elements of $\bigcup_{i=1}^{\infty} A_i$ according to their heights in increasing order without listing same element again as follows.

$$a_1^1, a_1^2, a_2^1, a_1^3, a_2^2, a_3^1, a_1^4, a_2^3, a_3^2, a_4^1, \ \ldots$$

This arrangement can be viewed as shown in the following figure where the elements with same height are placed on same diagonal.

This arrangement assures One-to-One correspondance of $\bigcup_{i=1}^{\infty} A_i$ with the set of

Table 1: Digonal arrangements of elements with same height.

a_1^1	a_1^2	a_1^3	a_1^4	a_1^5	$\ldots$
a_2^1	a_2^2	a_2^3	a_2^4	a_2^5	$\ldots$
a_3^1	a_3^2	a_3^3	a_3^4	a_3^5	$\ldots$
a_4^1	a_4^2	a_4^3	a_4^4	a_4^5	$\ldots$
a_5^1	a_5^2	a_5^3	a_5^4	a_5^5	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

natural numbers. Hence, $\bigcup_{i=1}^{\infty} A_i$ is countable.

7. Prove that the set of rational numbers is countable.

Proof:
We have the set of rationals defined by $Q = \left\{ \frac{p}{q} / p, q \in Z,\ q \neq 0 \right\}$
Now define

$$E_n = \left\{ \frac{0}{n}, \frac{-1}{n}, \frac{1}{n}, \frac{-2}{n}, \frac{2}{n}, \dots \right\}$$

Each E_n is countable as E_n has a one-to-one correspondance with the set of integers, hence with the set of positive integers.

Now,

$$Q = \bigcup_{i=1}^{\infty} E_i$$

Since, Q is a countable union of countable sets, it is countable.

> 8. **Prove that every infinite subset of a countable set is countable.**

Proof:
Let A be a countable set and B be an infinite subset of A. Suppose, $A = \{a_1, a_2, \dots\}$

Then each member of B is one of a_i.

Let n_1 be the smallest subscript for which a_{n_1} belongs to B.

Let n_2 be the next smallest subscript for which a_{n_2} belongs to B

Continuing similarly we shall be able to write the infintie set as follows

$$B = \{a_{n_1}, a_{n_2}, \dots\}$$

As the elements of B are labled with $1, 2, 3, \dots$ the set B is equivalent to the set of postitive integers. Hence, B is countable.

> 9. **Prove that set of all rationals in the interval $[0, 1]$ set is countable.**

Proof:
We know that the set of all the rational numbers Q is countable.

As there are infinitely many rational numbers between any two real numbers, we have infinitely many rational numbers in $[0, 1]$.

Thus, the set of rational numbers in $[0, 1]$ is an infinte subset of the countable set Q.

Since every infinite subset of a countable set is also countable, we conclude that the set of all rational numbers in $[0, 1]$ is countable.

10. Metric Space

Metric Space

Let M be a non-empty set. A function $\rho : M \times M \to [0, \infty)$ is said to be a **Metric** for M if the following four properties are satisfied

(i) $\rho(x, x) = 0; (x \in M)$,

(ii) $\rho(x, y) > 0 (x, y \in M, x \neq y)$,

(iii) $\rho(x, y) = \rho(y, x); (x, y \in M)$,

(iv) $\rho(x, y) \leq \rho(x, z) + \rho(z, y); (x, y, z \in M)$ (triangle inequality).

Along with the metric ρ the set M is known as a **Metric Space** which is generally denoted by (M, ρ)

11. Absolute Value Metric on R

Absolute Value Metric on $\mathbb{R}$

A matrix $\rho : R \times R \to [0, \infty]$ defined by

$$d(x, y) = |x - y|, \ \forall x, y \in R$$

is called the **Absolute Value Matric** on R.

12. Show that $\rho : R \times R \to R$, defined by $\rho(x, y) = |x - y|$, is a metric on R

Answer

Here, $\rho : R \times R \to R$ is defined by $\rho(x, y) = |x - y|$.

For $x, y, z \in R$

(i) $d(x, y) = |x - y| \geq 0$

(ii) $d(x, y) = 0 \iff |x - y| = 0 \iff x = y$

(iii) $d(x, y) = |x - y| = |y - x| = d(y, x)$

Therefore, for any x and y in $\mathbb{R}$ we have $d(x, y) = d(y, x)$

(iv) We have,

$$\begin{aligned} d(x, y) &= |x - y| \\ &= |x - z + z - y| \\ &\leqslant |x - z| + |z - y| \\ &= d(x, z) + d(z, y) \end{aligned}$$

Therefore, $d(x, y) \leqslant d(x, z) + d(z, y)$.

From (i), (ii), (iii) and (iv), it follows that d is a metric for R.

13. Discrete Metric on R

Discrete Metric
A function $d : R \times R \to R$ defined by

$$d(x,y) = \begin{cases} 1\ ; & \text{if } x \neq y \\ 0\ ; & \text{if } x = y \end{cases}$$

is called **Discrete Metric** for R. The discrete metric space (R, d) is also denoted by R_d

14. Show that $d : R \times R \longrightarrow R$ defined by

$$d(x,y) = \begin{cases} 1\ ; & \textbf{if } x \neq y \\ 0\ ; & \textbf{when } x = y \end{cases}$$

is a metric on R.

Answer:
Here $d : R \times R \longrightarrow R$ is defined by

$$d(x,y) = \begin{cases} 1 & ;\ \text{if } x \neq y \\ 0 & ;\ \text{if } x = y \end{cases}$$

For $x, y, z \in R$

1. By the definition of d we have,

$$d(x,y) = 0 \quad \text{or} \quad d(x,y) = 1$$

$$\therefore d(x,y) \geq 0$$

2. $d(x,y) = 0 \iff x = y$ (by the definition)

3. If $x = y$ than $d(x,y) = 0 = d(y,x)$
   and

   if $x \neq y$ then $d(x,y) = 1 = d(y,x)$
   thus for any x and y in R we have $d(x,y) = d(y,x)$

4. Now, if $x = y$ then we have $d(x, y) = 0$
   As $0 \leqslant d(x, z)$ and $0 \leqslant d(z, y)$ we get,

$$d(x, y) \leq d(x, z) + d(z, y)$$

   Also if $x \neq y$ then $d(x, y) = 1$.

   Moreover, any $z \in R$ must differ from atleast one of x and y. Therefore, $x \neq z$ or $y \neq z$.

$$\therefore \quad d(x, z) = 1 \text{ or } d(z, y) = 1$$

$$\therefore \quad d(x, y) = d(x, z) \quad \text{or} \quad d(x, y) = d(z, y)$$

   So, in this case also,

$$d(x, y) \leq d(x, z) + d(z, y)$$

   . From (1), (2), (3) and (4), it follows that d is a metric for R.

15. **Let $\rho : R^n \times R^n \longrightarrow R$ be defined as**

$$\rho(x, y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{\frac{1}{2}}$$

where $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in R^n$. Then prove that ρ is a metric space.

Proof

For $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n), z = (z_1, z_2, ..., z_n) \in R^n$,

1. $\rho(x, y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{\frac{1}{2}} \geqslant 0$

2.

$$\begin{aligned} \rho(x, y) = 0 &\iff \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{\frac{1}{2}} = 0 \\ &\iff (x_k - y_k)^2 = 0, \ \forall\ k = 1, 2, \ldots, n \\ &\iff x_k = y_k, \ \forall\ k = 1, 2, \ldots, n \\ &\iff (x_1, x_2, ..., x_n) = (y_1, y_2, ..., y_n) \end{aligned}$$

   $\therefore \rho(x, y) = 0 \iff x = y$

3.

$$\rho(x, y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{\frac{1}{2}} = \left[\sum_{k=1}^{n} (y_k - x_k)^2\right]^{\frac{1}{2}} = \rho(y, x)$$

4. For $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ and $z = (z_1, z_2, ..., z_n)$ let us define, $a_k = x_k - z_k$, $b_k = z_k - y_k$. Clearly, $a_k + b_k = x_k - y_k$. Now,

$$\begin{aligned}
\rho(x,y) &= \left[\sum_{k=1}^{n}(x_k - y_k)^2\right]^{\frac{1}{2}} \\
&= \left[\sum_{k=1}^{n}(a_k + b_k)^2\right]^{\frac{1}{2}} \\
&\leqslant \left[\sum_{k=1}^{n} a_k^2\right]^{\frac{1}{2}} + \left[\sum_{k=1}^{n} b_k^2\right]^{\frac{1}{2}} \quad \text{(By Minkowski's inequality)} \\
&\leqslant \left[\sum_{k=1}^{n}(x_k - z_k)^2\right]^{\frac{1}{2}} + \left[\sum_{k=1}^{n}(z_k - y_k)^2\right]^{\frac{1}{2}} \\
&= \rho(x,z) + \rho(z,k)
\end{aligned}$$

Thus we have, $\rho(x,y) \leqslant \rho(x,z) + \rho(z,y)$

From (1), (2), (3) and (4), it follows that ρ is a metric for R^n

16. **For $P(x_1, y_1)$ and $Q(x_2, y_2)$ in R^2, define $\sigma : R^2 \times R^2 \longrightarrow R$ by**

$$\sigma(P,Q) = |x_1 - x_2| + |y_1 - y_2|$$

, show that σ is a metric on R^2

Answer:

For $P(x_1, y_1)$,$Q(x_2, y_2)$ and $S(x_3, y_3)$ in R^2

1. $\sigma(P,Q) = |x_1 - x_2| + |y_1 - y_2| \geqslant 0$
2. $$\begin{aligned}
\sigma(P,Q) = 0 &\iff |x_1 - x_2| + |y_1 - y_2| = 0 \\
&\iff |x_1 - x_2| = 0, |y_1 - y_2| = 0 \\
&\iff x_1 = x_2, y_1 = y_2 \\
&\iff (x_1, y_1) = (x_2, y_2) \\
&\iff P = Q
\end{aligned}$$
3. $\sigma(P,Q) = |x_1 - x_2| + |y_1 - y_2| = |x_2 - x_1| = 0, |y_2 - y_1| = \sigma(Q,P)$
4. $$\begin{aligned}
\sigma(P,Q) &= |x_1 - x_2| + |y_1 - y_2| \\
&= |x_1 - x_3 + x_3 - x_2| + |y_1 - y_3 + y_3 - y_2| \\
&\leqslant |x_1 - x_3| + |x_3 - x_2| + |y_1 - y_3| + |y_3 - y_2| \\
&\leqslant (|x_1 - x_3| + |y_1 - y_3|) + (|x_3 - x_2| + |y_3 - y_2|)\,| \\
&= \sigma(P,S) + \sigma(S,Q)
\end{aligned}$$

Therefore, $\sigma(P,Q) \leqslant \sigma(P,Q) + \sigma(S,Q)$

From (1), (2), (3) and (4), it follows that σ is a metric for R^2

> **17. Let (M,d) be a metric space and let $d^*(x,y) = min\{1, d(x,y)\}$. Then prove that d^* is a metric on M**

Proof:
By the definition, $d^*(x,y) = min\{1, d(x,y)\}$

Therefore,

$$d^*(x,y) \leqslant 1 \quad \text{and} \quad d^*(x,y) \leqslant d(x,y)$$

Now, for $x, y, z \in R$

1. As $d^*(x,y) = min\{1, d(x,y)\}$ and $d(x,y) \geqslant 0$, we have, $d^*(x,y) \geq 0$
2. $d^*(x,y) = 0 \iff min\{1, d(x,y)\} = 0 \iff d(x,y) = 0 \iff x = y$
3. If $d^*(x,y) = min\{1, d(x,y)\} = min\{1, d(y,x)\} = d^*(y,x)$
4. First, suppose $d^*(x,z) = 1$ or $d^*(z,y) = 1$ or both are equal to 1 then as $d^*(x,y) \leqslant 1$ we get

   $$d^*(x,y) \leqslant d^*(x,z) \quad \text{or} \quad d^*(x,y) \leqslant d^*(z,y)$$

   Therefore,

   $$d^*(x,y) \leqslant d^*(x,z) + d^*(z,y)$$

   Next, suppose $d^*(x,z) \neq 1$ and $d^*(z,y) \neq 1$
   Then, $d^*(x,z) = d(x,z)$ and $d^*(z,y) = d(z,y)$
   As $d^*(x,y) \leqslant d(x,y)$ and $d(x,y) \leqslant d(x,z) + d(z,y)$, we get,

   $$d^*(x,y) \leqslant d(x,z) + d(z,y)$$

   Therefore,

   $$d^*(x,y) \leqslant d^*(x,z) + d^*(z,y)$$

   Thus, in any case we have,

   $$d^*(x,y) \leqslant d^*(x,z) + d^*(z,y)$$

   From (1), (2), (3) and (4), it follows that d^* is a metric for R.

> **18. Let (M,d) be a metric space and let $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$. Then prove that d_1 is a metric on M**

Proof:
For a metric space (M, d), a function d_1 is defined by

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

. Now, for $x, y, z \in M$

1. $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)} \geqslant 0$

2. $d_1(x,y) = 0 \iff \dfrac{d(x,y)}{1+d(x,y)} = 0 \iff d(x,y) = 0 \iff x = y$

3. $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)} = \dfrac{d(y,x)}{1+d(y,x)} = d_1(y,x)$

4. Also, as d is a metric for M, for $z \in M$, we have

$$d(x,y) \leqslant d(x,z) + d(z,y)$$

Adding $d(x,y)[d(x,z)+d(z,y)]$ on both the sides, we get,

$$\begin{aligned}
\therefore d(x,y) + d(x,y)[d(x,z)+d(z,y)] &\leqslant [d(x,z)+d(z,y)] + d(x,y)[d(x,z)+d(z,y)] \\
\therefore d(x,y)[1+d(x,z)+d(z,y)] &\leqslant [d(x,z)+d(z,y)](1+d(x,y)) \\
\therefore \frac{d(x,y)}{1+d(x,y)} &\leqslant \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \\
\therefore \frac{d(x,y)}{1+d(x,y)} &\leqslant \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)} \\
\therefore \frac{d(x,y)}{1+d(x,y)} &\leqslant \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\
\therefore d_1(x,y) &\leqslant d_1(x,z) + d_1(z,y)
\end{aligned}$$

From (1),(2),(3) and (4) it follows that d_1 is also a metric on M.

19. Show that if ρ is a metric on a set M then so is 2ρ

Answer:
Here, ρ is a metric on M. For all $x, y \in M$ function 2ρ is defined by

$$(2\rho)(x,y) = 2\rho(x,y)$$

Now, for $x, y, z \in M$

1. $(2\rho)(x,y) = 2\rho(x,y) \geqslant 0 \qquad (\because \rho(x,y) \geqslant 0)$
2. $(2\rho)(x,y) = 0 \iff 2\rho(x,y) = 0 \iff \rho(x,y) = 0 \iff x = y$

3. $(2\rho)(x,y) = 2\rho(x,y) = 2\rho(y,x) = (2\rho)(y,x)$

4.

$$\begin{aligned}(2\rho)(x,y) &= 2\rho(x,y)\\ &\leqslant 2[\rho(x,z)+\rho(z,y)]\\ &\leqslant 2\rho(x,z)+2\rho(z,y)\\ &\leqslant (2\rho)(x,z)+(2\rho)(z,y)\end{aligned}$$

Therefore, $(2\rho)(x,y) \leqslant (2\rho)(x,z)+(2\rho)(z,y)$ From (1),(2),(3) and (4) it follows that 2ρ is also a metric on M.

20. Show that if ρ and σ are metrics on a set M then $\rho+\sigma$ is a metric on M.

Answer:
Here, ρ and σ are metrics on M. For all $x,y \in M$ function $\rho+\sigma$ is defined by

$$(\rho+\sigma)(x,y) = \rho(x,y)+\sigma(x,y)$$

Now, for $x,y,z \in M$

1. $(\rho+\sigma)(x,y) = \rho(x,y)+\sigma(x,y) \geqslant 0 \qquad (\because \rho(x,y)\geqslant 0, \sigma(x,y)\geqslant 0)$

2. $(\rho+\sigma)(x,y) = 0 \iff \rho(x,y)+\sigma(x,y)=0 \iff \rho(x,y)=0, \sigma(x,y)=0 \iff x=y$

3. $(\rho+\sigma)(x,y) = \rho(x,y)+\sigma(x,y) = \rho(y,x)+\sigma(y,x) = (\rho+\sigma)(y,x)$

4.

$$\begin{aligned}(\rho+\sigma)(x,y) &= \rho(x,y)+\sigma(x,y)\\ &\leqslant [\rho(x,z)+\rho(z,y)]+[\sigma(x,z)+\sigma(z,y)]\\ &= [\rho(x,z)+\sigma(x,z)]+[\rho(z,y)+\sigma(z,y)]\\ &= (\rho+\sigma)(x,z)+(\rho+\sigma)(z,y)\end{aligned}$$

Therefore, $(\rho+\sigma)(x,y) \leqslant (\rho+\sigma)(x,z)+(\rho+\sigma)(z,y)$

From (1),(2),(3) and (4) it follows that $\rho+\sigma$ is also a metric on M.

21. Let $d : R \times R \longrightarrow R$ be defined by $d(x,y) = \sin|x-y|$. Check whether d is a metric or not.

Answer:
For $2\pi, \pi \in R$, we have
$d(2\pi,\pi) = \sin|2\pi-\pi| = \sin\pi = 0$, but $2\pi \neq \pi$.
$\therefore$ d is not a metric on R

22. **Let $d : [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \longrightarrow R$ be defined by $d(x, y) = \sin|x - y|$. Show that d is a metric on $[0, \frac{\pi}{2}]$.**

Answer:
For $x, y, z \in M$

1. For, $x, y \in [0, \frac{\pi}{2}]$ we have $\leqslant |x - y| \leqslant \frac{\pi}{2}$
   Therefore, $d(x, y) = \sin|x - y| \geqslant 0$
2. $d(x, y) = 0 \iff \sin|x - y| = 0 \iff |x - y| = 0 \iff x = y$
3. $d(x, y) = \sin|x - y| = \sin|y - x| = d(y, x)$
4. Now, for any $x, y, z \in [0, \frac{\pi}{2}]$ we have $\leqslant |x - y| \leqslant \frac{\pi}{2}$, $\leqslant |x - z| \leqslant \frac{\pi}{2}$ and $\leqslant |z - y| \leqslant \frac{\pi}{2}$ Also,

$$\begin{aligned} d(x, y) &= \sin|x - y| \\ &= |\sin(x - y)| \qquad\qquad = |\sin(x - z + z - y)| \\ &= |\sin(x - z)\cos(z - y) + \cos(x - z)\sin(z - y) \\ &\leqslant |\sin(x - z)|\cos(z - y)| + |\cos(x - z)|\sin(z - y)| \\ &\leqslant \sin|x - z| + \sin|z - y| (\because |\cos\theta| \leqslant | \\ &= d(x, z) + d(z, y) \end{aligned}$$

   Therefore, $d(x, y) \leqslant d(x, z) + d(z, y)$

From (1),(2),(3) and (4) it follows that d is also a metric on $[0, \frac{\pi}{2}]$.

23. **If $d : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $d(x, y) = |x^2 - y^2|$ then check whether d is a metric or not.**

Answer:
For $2, -2 \in \mathbb{R}$, we have
$d(2, -2) = |2^2 - (-2)^2| = 0$, but $2 \neq -2$.
$\therefore$ d is not a metric on $\mathbb{R}$

24. **Cluster point.**

Cluster Point:
Let (M, ρ) be a metric space and $A \subset M$. A point $x \in M$ is said to be a cluster point of A if for each $h > 0$ there is some $y \in A$ such that

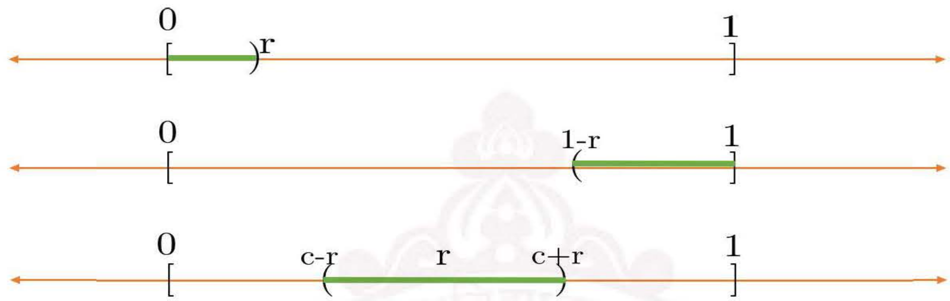
$$0 < \rho(x, y) < h$$

25. **Show that the set of all cluster points of $(0, 1)$ is $[0, 1]$**

Answer:
Here, $(0, 1)$ is to be treated as a subset of R^1 (i.e. R with absolute value matrix)

First we show that each point of $[0, 1]$ is a cluster point of $(0, 1)$

For any $r > 0$, open interval $(0, r)$, intersects $(0, 1)$ at infinitely many points. So 0 is a cluster point of $(0, 1)$. Also, for any $r > 0$, $(1 - r, 1)$, intersects $(0, 1)$ at infinitely many points. So 1 is a cluster point of $(0, 1)$. For any $r > 0$, and $c \in (0, 1)$, the interval $(c-r, c+r)$, intersects



$(0, 1)$ at infinitely many points. So c is a cluster point of $(0, 1)$. Thus, each point in $[0, 1]$ is a cluster point of $(0, 1)$

Finally, we show that no point out side $[0, 1]$ can be a cluster point of $(0, 1)$

Let $x \notin [0, 1]$.

If $1 < x$ then we can choose some sufficiently small $\epsilon > 0$ so that

$$1 < x - \epsilon < x$$

Therefore,

$$(0, 1) \cap (x - \epsilon, x + \epsilon) = \emptyset$$

So, if choose some r such thth $0 < r < x - \epsilon$ then for every $y \in (0, 1)$ we have,

$$r < |x - y|$$

So, x cannot be a cluster point of $(0, 1)$.

Similarly it can be shown that if $x < 0$ then also x cannot be a cluster point of $(0, 1)$ the set of all cluster points of $(0, 1)$ is $[0, 1]$

26. Find the cluster points of

[A] (a, b)

Answer: $[a, b]$ Proof similar to that given for the set of limit points of $(0, 1)$.

[B] $[a, b]$

Answer: $[a, b]$ Proof similar to that given for the set of limit points of $(0, 1)$.

[C] Q

Answer:
As every neighbourhood of any real number contains infinitely many rational numbers, all real numbers are cluster points of Q. Therefore for the set of cluster points of Q is R.

[D] $R\backslash Q$

Answer:
Here, $R\backslash Q$ is the set of all irrational numbers.

As every neighbourhood of any real number contains infinitely many irrational numbers, all real numbers are cluster points of $R\backslash Q$. Therefore for the set of cluster points of $R\backslash Q$ is R.

[E] R

As every neighbourhood of any real number contains infinitely many real numbers, all real numbers are cluster points of R.

[F] N

Distance between two consecutive integers is 1.

Therefore. for any real number x we can always find some sufficiently small $\delta > 0$ such that

$$(x - \delta, x + \delta)$$

contains no positive integer other than, possibly x.

Hence, no real number is a cluster point of N.

Therefore, the set of cluster points of N is $\emptyset$.

[G] Z

Distance between two consecutive integers is 1.

Therefore. for any real number x we can always find some sufficiently small $\delta > 0$ such that

$$(x-\delta, x+\delta)$$

contains no integer other than, possibly x.

Hence, no real number is a cluster point of Z.

Therefore, the set of cluster points of Z is $\emptyset$.

[H] $\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\}$

For any given $\epsilon > 0$, by Archimedean Property of R^1, there is some positive integer n such that

$$1 < n\epsilon$$

Therefore,

$$\frac{1}{n} < \epsilon$$

Thus, for any $\epsilon > 0$ there is some positive integer n such that

$$\frac{1}{n} \in (-\epsilon, \epsilon)$$

Therefore 0 is a cluster point of the given set.

Also, for any other $x \neq 0$ we can find sufficiently small $\delta > 0$ such that

$$(x-\delta, x+\delta)$$

contains no member of the sequence other than x. The set of cluster points of $\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\}$ is $\{0\}$

27. Limit of a function

Limit of a function

Let (M_1, ρ_1) and (M_2, ρ_2) be two metric spaces and $a \in M_1$. Also let $f : M_1 \longrightarrow M_2$ be a function and $L \in M_2$. Then L is said to be limit of f as x tends to a, if for each $\epsilon > 0$ there exists some $\delta > 0$ such that

$$\rho_2(f(a), f(x)) < \epsilon \quad \text{whenever} \quad 0 < \rho_1(a, x) < \delta$$

28. Let (M, ρ) be a metric space and let a be a point in M. Let f and g be real valued functions whose domains are subsets of M. If $\lim\limits_{x\longrightarrow a} f(x) = L$ and $\lim\limits_{x\longrightarrow a} g(x) = N$ then prove that

[A] $\lim_{x\longrightarrow a}[f(x)+g(x)] = L+N$

Proof:
Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = N$
Then, for any given $\epsilon > 0$ there exist some $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - L| < \frac{\epsilon}{2}, \quad \text{whenever } 0 < \rho(x,a) < \delta_1 ---(1)$$

$$|g(x) - N| < \frac{\epsilon}{2}, \quad \text{whenever } 0 < \rho(x,a) < \delta_2 ---(2)$$

If we take $\delta = \min\{\delta_1, \delta_2\}$ then for $0 < \rho(x,a) < \delta$, (1) and (2) hold true.

Therefore, for $0 < \rho(x,a) < \delta$ we have,

$$\begin{aligned} |(f(x)+g(x)) - (L+N)| &= |(f(x)-L) + (g(x)-N)| \\ &< |f(x)-L| + |g(x)-N| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ whenever } 0 < \rho(x,a) < \delta \\ &= \epsilon \end{aligned}$$

$$\therefore |(f(x)+g(x)) - (L+N)| < \epsilon \quad \text{whenever } 0 < \rho(x,a) < \delta$$

Hence,

$$\lim_{\mathbf{x}\to\mathbf{a}}[\mathbf{f(x)}+\mathbf{g(x)}] = \mathbf{L}+\mathbf{N}$$

[B] $\lim_{x\longrightarrow a}[f(x)-g(x)] = L-N$

Proof:
Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = N$
Then, for any given $\epsilon > 0$ there exist some $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - L| < \frac{\epsilon}{2}, \quad \text{whenever } 0 < \rho(x,a) < \delta_1 ---(1)$$

$$|g(x) - N| < \frac{\epsilon}{2}, \quad \text{whenever } 0 < \rho(x,a) < \delta_2 ---(2)$$

If we take $\delta = \min\{\delta_1, \delta_2\}$ then for $0 < \rho(x,a) < \delta$, (1) and (2) hold true.

Therefore, for $0 < \rho(x,a) < \delta$ we have,

$$\begin{aligned} |(f(x)-g(x)) - (L-N)| &= |(f(x)-L) + (N-g(x))| \\ &\leqslant |f(x)-L| + |N-g(x)| \\ &= |f(x)-L| + |g(x)-N| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ whenever } 0 < \rho(x,a) < \delta \\ &= \epsilon \end{aligned}$$

$$\therefore |(f(x)-g(x)) - (L-N)| < \epsilon \quad \text{whenever } 0 < \rho(x,a) < \delta$$

Hence,

$$\lim_{\mathbf{x}\to\mathbf{a}}[\mathbf{f(x)} - \mathbf{g(x)}] = \mathbf{L} - \mathbf{N}$$

[C] $\lim\limits_{x\longrightarrow a}[f(x).g(x)] = \lim\limits_{x\to a} f(x). \lim\limits_{x\to a} g(x)$

Proof:

Let $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = N$

Now,

$$\begin{aligned}|f(x).g(x) - LN| &= |f(x).g(x) - L.g(x) + L.g(x) - LN| \\ &\leqslant |g(x)||f(x) - L| + |L||g(x) - N|\end{aligned}$$

Therefore,

$$|f(x).g(x) - LN| \leqslant |g(x)||f(x) - L| + |L||g(x) - N| \quad \text{- - - (1)}$$

As $\lim\limits_{x\to a} g(x) = N$, for 1 there must be some $\delta_1 > 0$ such that

$$|g(x) - N| < 1, \quad \text{whenever } 0 < \rho(x,a) < \delta_1$$

Now,

$$\begin{aligned}|g(x)| &= |g(x) - N + N| \\ &\leqslant |g(x) - N| + |N| \\ &< 1 + |N|, \quad \text{whenever } \; 0 < \rho(x,a) < \delta_1\end{aligned}$$

Therefore,

$$|g(x)| < |N| + 1, \quad \text{whenever } 0 < \rho(x,a) < \delta_1$$

Hence, for $0 < \rho(x,a) < \delta_1$ from (1), we have ,

$$|f(x).g(x) - LN| < (|N| + 1)|f(x) - L| + |L||g(x) - N| - - - (2)$$

Again considering $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = N$, for any given $\epsilon > 0$ there must be some $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$|f(x) - L| < \frac{\epsilon}{2(|N| + 1)}, \quad \text{whenever } 0 < \rho(x,a) < \delta_2 - - - (3)$$

$$|g(x) - N| < \frac{\epsilon}{2(|L| + 1)}, \quad \text{whenever } 0 < \rho(x,a) < \delta_3 - - - (4)$$

If we take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ then all of (2),(3) and (4) hold true with each δ_1, δ_2 and δ_3 replaced by δ.

Therefore, for $0 < \rho(x,a) < \delta$, we have,

$$
\begin{aligned}
|f(x).g(x) - LN| &< (|N|+1)|f(x) - L| + |L||g(x) - N| \\
&< (|N|+1)\frac{\epsilon}{2(|N|+1)} + |L|\frac{\epsilon}{2(|L|+1)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
\therefore |f(x).g(x) - LN| &< \epsilon \quad \text{whenever } 0 < \rho(x,a) < \delta
\end{aligned}
$$

Hence,

$$\boxed{\lim_{\mathbf{x}\to\mathbf{a}}[\mathbf{f(x)g(x)}] = \mathbf{LN}}$$

[D] $\displaystyle\lim_{x\longrightarrow a}\frac{f(x)}{g(x)} = \frac{L}{N}$ **if** $N \neq 0$

<u>Proof:</u>

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = N,$ where $N \neq 0$

As, $N \neq 0$ implies that $|N| > 0$.

Therefore, as $\lim_{x\to a} g(x) = N$, for $\frac{|N|}{2} > 0$ there exists some $\delta_1 > 0$ such that

$$|g(x) - N| < \frac{|N|}{2}, \quad \text{whenever } 0 < \rho(x,a) < \delta_1$$

Now,

$$
\begin{aligned}
|N| &= |N - g(x) + g(x)| \\
\therefore |N| &\leqslant |N - g(x)| + |g(x)| \\
\therefore |N| &\leqslant |g(x) - N| + |g(x)| \\
\therefore |N| &< \frac{|N|}{2} + |g(x)|, \quad \text{whenever } \; 0 < \rho(x,a) < \delta_1 \\
\therefore |N| - \frac{|N|}{2} &< |g(x)| \\
\therefore \frac{|N|}{2} &< |g(x)|
\end{aligned}
$$

Hence,

$$\frac{1}{|g(x)|} < \frac{2}{|N|}, \quad \text{whenever } 0 < \rho(x,a) < \delta_1 \text{ - - - } (1)$$

Now,

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{L}{N}\right| &= \left|\frac{f(x).N - L.g(x)}{N.g(x)}\right| \\
&< \frac{2}{|N|^2}|f(x).N - L.g(x)| \quad \text{whenever } 0 < \rho(x,a) < \delta_1 \quad \text{(from (1))} \\
&< \frac{2}{|N|^2}|f(x).N - LN + LN - L.g(x)| \quad \text{whenever } 0 < \rho(x,a) < \delta_1 \\
&< \frac{2}{|N|^2}|N||f(x) - L| + \frac{2|L|}{N^2}|g(x) - N| \quad \text{whenever } 0 < \rho(x,a) < \delta_1 \\
&< \frac{2}{|N|}|f(x) - L| + \frac{2|L|}{N^2}|g(x) - N| \quad \text{whenever } 0 < \rho(x,a) < \delta_1
\end{aligned}$$

Hence, for $0 < \rho(x,a) < \delta_1$ from (1), we have ,

$$\left|\frac{f(x)}{g(x)} - \frac{L}{N}\right| < \frac{2}{|N|}|f(x) - L| + \frac{2|L|}{N^2}||g(x) - N| - - - (2)$$

Again, considering $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = N$, for any given $\epsilon > 0$ there exist some $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$|f(x) - L| < \frac{|N|\epsilon}{4}, \quad \text{whenever } 0 < \rho(x,a) < \delta_2 - - - (3)$$

$$|g(x) - N| < \frac{N^2\epsilon}{4(|L|+1)}, \quad \text{whenever } 0 < \rho(x,a) < \delta_3 - - - (4)$$

If we take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ then all of (2),(3) and (4) hold true with each of δ_1, δ_2 and δ_3 replaced by δ.
Therefore, for $0 < \rho(x,a) < \delta$, we have,

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{L}{N}\right| &< \frac{2}{|N|}|f(x) - L| + \frac{2|L|}{N^2}|g(x) - N| \\
&< \frac{2}{|N|}\left(\frac{|N|\epsilon}{4}\right) + \frac{2|L|}{N^2}\left(\frac{N^2\epsilon}{4(|L|+1)}\right) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
\therefore \left|\frac{f(x)}{g(x)} - \frac{L}{N}\right| &< \epsilon \quad \text{whenever } 0 < \rho(x,a) < \delta
\end{aligned}$$

Hence,

$$\boxed{\lim_{\mathbf{x} \to \mathbf{a}} \frac{\mathbf{f(x)}}{\mathbf{g(x)}} = \frac{\mathbf{L}}{\mathbf{N}}}$$

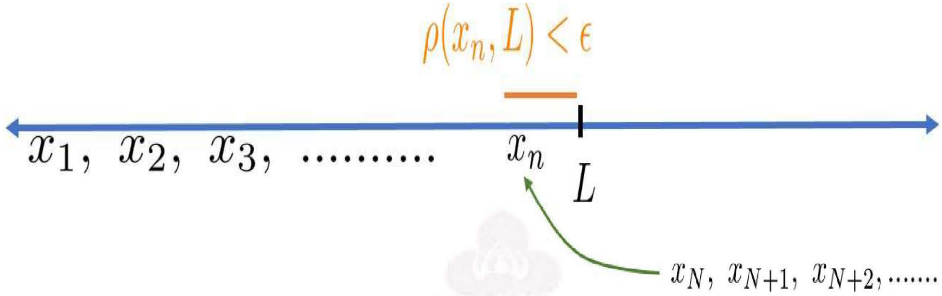
29. Convergence of a sequence

Convergence of a sequence:

Let (M, ρ) be a metric space and $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in M. If $L \in M$ is point in M such that for every $\epsilon > 0$ there exists some positive integer N such that

$$\rho(s_n, L) < \epsilon, \quad \text{whenever} \quad n \geqslant N$$

then L is said to be the limit of sequence $\{x_n\}_{n=1}^{\infty}$ as n tends to ∞. In symbols it is written as



$$\lim_{n \longrightarrow \infty} s_n = L$$

In this case we say that $\{x_n\}_{n=1}^{\infty}$ is **Convergent** in M to a point L.

30. Cauchy sequence

Cauchy sequence:

A sequence $\{x_n\}_{n=1}^{\infty}$ of points in a metric space (M, ρ) is said to be a **Cauchy Sequence** if for a given $\epsilon > 0$, there exists some $N \in I$ such that

$$\rho(x_m, x_n) < \epsilon \qquad , \forall\, m, n \geqslant N$$

.

31. Prove that if $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of points in a metric space (M, ρ) then $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Is the converse true? Justify.

Proof:

Let $\{s_n\}$ be a sequence of points in a metric space (M, ρ)

Suppose, $\{s_n\}$ converges to a point l in M.

Therefore, For any given $\epsilon > 0$ there exists a positive number N such that

$$\rho(s_n, l) < \frac{\epsilon}{2}, \quad \text{whenever} \quad n \geqslant N$$

Therefore, for any two positive integers $n_1 > N$ and $n_2 > N$ we have,

$$\rho(s_{n_1}, l) < \frac{\epsilon}{2} \quad \text{and} \quad \rho(s_{n_2}, l) < \frac{\epsilon}{2}$$

Therefore, for $n_1 > N$ and $n_2 > N$, we get,

$$\begin{aligned}\rho(s_{n_1}, s_{n_2}) &< \rho(s_{n_1}, l) + \rho(s_{n_2}, l)\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}\\ &= \epsilon\end{aligned}$$

Hence, $\{s_n\}$ is a Cauchy sequence.

The converse is not true.

Justification
Consider the sequence $\{\frac{1}{n}\}$ of points in $(0,1)$. This is a Cauchy sequence in $(0,1)$.

Since, it converges to 0 in R but

$$0 \notin (0,1)$$

the sequence does not converge to a point in $(0,1)$

Therefore, every Cauchy sequence is not necessarily convergent.

32. **Show that a sequence in a metric space cannot convergene to two distinct limits.**

Answer:
Suppose, a sequence $\{a_n\}$ of points in a metric space (M, ρ) converges to two limits l_1 and l_2. Therefore, for any given $\epsilon > 0$ there are some positive integers N_1 and N_2 such that

$$\rho(a_n, l_1) < \frac{\epsilon}{2} \text{ whenever } n \geqslant N_1$$

and

$$\rho(a_n, l_2) < \frac{\epsilon}{2} \text{ whenever } n \geqslant N_2$$

Therefore for

$$N = \max\{N_1, N_2\}$$

we have,

$$\rho(a_n, l_1) < \frac{\epsilon}{2} \quad \text{and} \quad \rho(a_n, l_2) < \frac{\epsilon}{2} \text{ whenever } n \geqslant N$$

For $n \geqslant N$ we get,

$$\begin{aligned}\rho(l_1, l_2) &\leqslant \rho(a_n, l_1) + \rho(a_n, l_2)\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}\\ &= \epsilon\\ \therefore \rho(l_1, l_2) &< \epsilon\end{aligned}$$

As $\rho(l_1, l_2)$ is less than every positive number we must have $\rho(l_1, l_2) = 0$.
Hence,

$$l_1 = l_2$$

. Thus, a sequence in a metric space cannot convergene to two distinct limits.

33. Show that if $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in R_d then there exists a positive integer N such that $x_N = x_{N+1} = x_{N+2} = ...$

Answer:
Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence of points in R_d (i.e. in (R, d)).

As the sequence is a convergent sequence it is a *Cauchy* sequence.
Therefore, for $\frac{1}{2} > 0$ there is some positive integer N such that

$$d(x_n, x_{n+p}) < \frac{1}{2}, \quad \forall\, n \geqslant N, p \geqslant 1$$

Now,

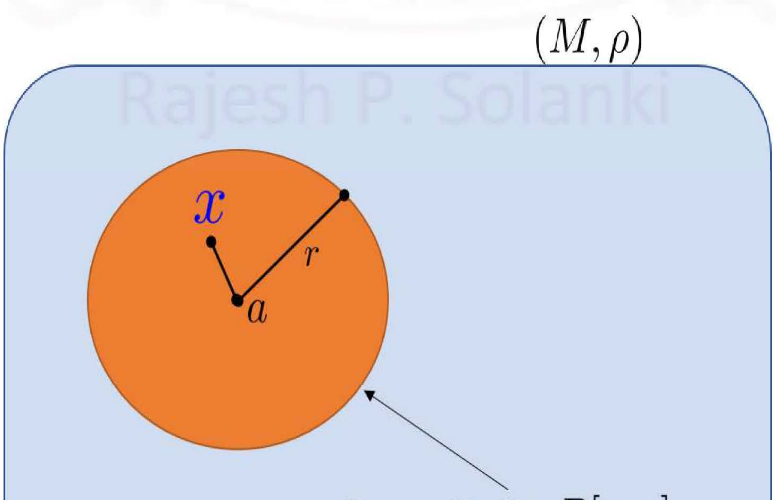
$$d(x_n, x_{n+p}) < \frac{1}{2} \Rightarrow d(x_n, x_{n+p}) = 0$$

But, then

$$x_n = x_{n+p}, \quad \forall\, n \geqslant N, p \geqslant 1$$
$$\therefore \quad x_N = x_{N+1} = x_{N+2} = ...$$

34. Open Ball

Open Ball:
Let (M, ρ) be a metric space. For $a \in M$ and $r > 0$ the set $\{x \in M /\ \rho(a, x) < r\}$ is called an



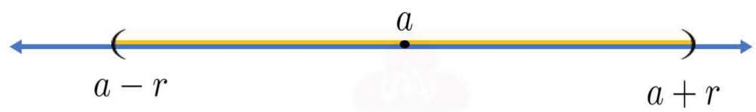
Open Ball centred at a with radius r and it is generally denoted by $B[a,r]$ or $B[a;r]$. Thus,

$$B[a,r] = \{x \in M/\rho(a,x) < r\}$$

Remark:
In R^1 (i.e. R with absolute value matrix) for $a \in R$ and $r > 0$ we have

$$B[a,r] = \{x \in R/|x-a| < r\}$$

Therefore, **an open ball in R^1 is an open interval**



$$B[a,r] = (a-r, a+r)$$

35. Continuity of a function

Continuity of a function:
Let (M_1, ρ_1) and (M_2, ρ_2) be two metric spaces. A function $f : M_1 \to M_2$ is said to be continuous at a point $a \in M_1$ if

$$\lim_{x \to a} f(x) = f(a)$$

Remark:
(1) From the definition it follows that,

$f : M_1 \to M_2$ is continuous at $a \in M_1$ *iff* for each $\epsilon > 0$ there exists some $\delta > 0$ such that

$$\rho_2(f(x), f(a)) < \epsilon \quad \text{whenever} \quad \rho_1(a,x) < \delta$$

(2) If we have $f : R^1 \to R^1$ then f is continuous at $a \in R_1$ *iff* for each $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \quad \text{whenever} \quad |x-a| < \delta$$

36. Prove that a real valued function f is continuous at $a \in R^1$ iff the inverse image under f of any open ball $B[f(a), \epsilon]$ about $f(a)$ contains an open ball $B[a, \delta]$ about a

Proof:
We know that,

$$f : R^1 \to R^1 \text{ is continuous at } a \in R^1 \iff \lim_{x \to a} f(x) = f(a)$$

Also,

$$\lim_{x\to a} f(x) = f(a) \iff \text{for any given } \epsilon > 0 \ \exists \text{ some } \delta > 0 \text{ such that}$$

$$|f(x) - f(a)| < \epsilon \ \text{ whenever } \ |x - a| < \delta$$

$$\iff f(x) \in (f(a) - \epsilon, f(a) + \epsilon) \ \text{ whenever } \ x \in (a - \delta, a + \delta)$$

$$\iff f(x) \in B[f(a), \epsilon] \ \text{ whenever } \ x \in B[a, \delta]$$

$$\iff f(B[a, \delta]) \subset B[f(a), \epsilon]$$

$$\iff B[a, \delta] \subset f^{-1}(B[f(a), \epsilon])$$

Hence, we conclude that a function f is continuous at $a \in R^1$ *iff* for every $\epsilon > 0$ there exists some $\delta > 0$ such that

$$f^{-1}(B[f(a), \epsilon]) \supset B[a, \delta]$$

37. **prove that a real valued function f is continuous at $a \in R$ iff whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers converging to a then $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$.**

Proof:

Let us first assume that f is continuous at a and prove that

$$\lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = f(a)$$

Since f is continuous at a for any given $\epsilon > 0$ there exists $\delta > 0$ such that

$$f(x) \in B[f(a), \epsilon] \ \text{ whenever } \ x \in B[a, \delta] - - - -(1)$$

Now, suppose $\{x_n\}_{n=1}^{\infty}$ is any sequence of real numbers converging to a. i.e.

$$\lim_{n\to\infty} x_n = a$$

Therefore, for $\delta > 0$ there is some $N \in I$ such that,

$$|x_n - a| < \delta, \quad \text{whenever} \quad n \geqslant N$$

$$\therefore \ x_n \in B[a, \delta], \quad \text{whenever} \quad n \geqslant N \text{ - - - } (2)$$

Taking $x = x_n$ in (1), it follows from (1) and (2) that,

$$f(x_n) \in B[f(a), \epsilon] \ \text{ whenever } \ n \geqslant N$$

Therefore

$$\lim_{n\to\infty} f(x_n) = f(a)$$

Thus,

$$\lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = f(a)$$

Conversely, suppose

$$\lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = f(a) \quad \text{- - - (3)}$$

If possible, suppose f is not continuous at a.
Therefore, there exists some $\epsilon > 0$ such that the inverse image under f of $B[f(a);\epsilon]$ contains no open ball about a.

Therefore, for each positive integer n we have

$$B\left[a, \frac{1}{n}\right] \not\subset f^{-1}(B[f(a),\epsilon])$$

Therefore for each $n \in I$ there must be some $x_n \in R$ such that

$$x_n \in B\left[a, \frac{1}{n}\right] \quad \text{but} \quad x_n \notin f^{-1}(B[f(a),\epsilon])$$

It implies that,

$$x_n \in B\left[a, \frac{1}{n}\right] \quad \text{but} \quad f(x_n) \notin B[f(a),\epsilon]$$

Thus, get a sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ such that,

$$|x_n - a| < \frac{1}{n} \quad \text{but} \quad |f(x_n) - f(a)| \geqslant \epsilon$$

. Since $\lim_{n\to\infty} \frac{1}{n} = 0$ it is clear that

$$\lim_{n\to\infty} x_n = a$$

But $|f(x_n) - f(a)| \geqslant \epsilon$ implies that

$$\lim_{n\to\infty} f(x_n) \neq f(a)$$

This is a contradiction to our assumption at (3). So, our supposition is wrong.

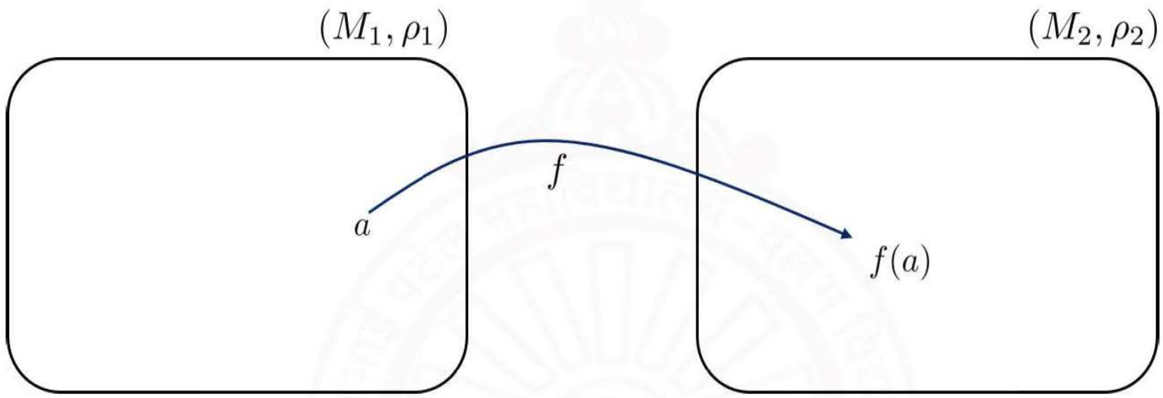
Hence, f must be continuous at a.

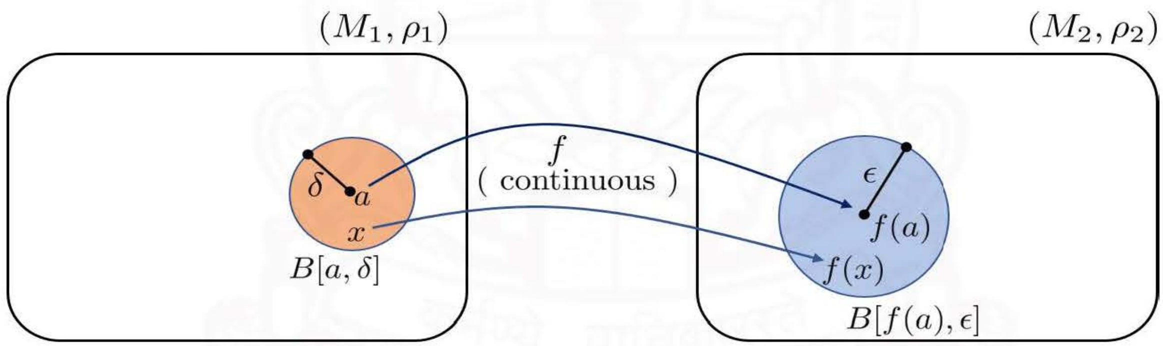
38. Let (M_1,ρ_1) and (M_2,ρ_2) be two metric spaces and $f : M_1 \to M_2$ be a function and $a \in M_1$. Then prove that a function f is said to be continuous at a iff one (and hence all) of the following conditions hold true.
(a) Given $\epsilon > 0$, $\exists\ \delta > 0$, $\ni \rho_2[f(x), f(a)] < \epsilon$ whenever $\rho_1(x,a) < \delta$
(b) The inverse image under f of any open ball $B[f(a),\epsilon]$ about $f(a)$ contains an open ball $B[a,\delta]$ about a.
(c) Whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in M_1 converging to a , then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ of points in M_2 converges to $f(a)$.

Proof:

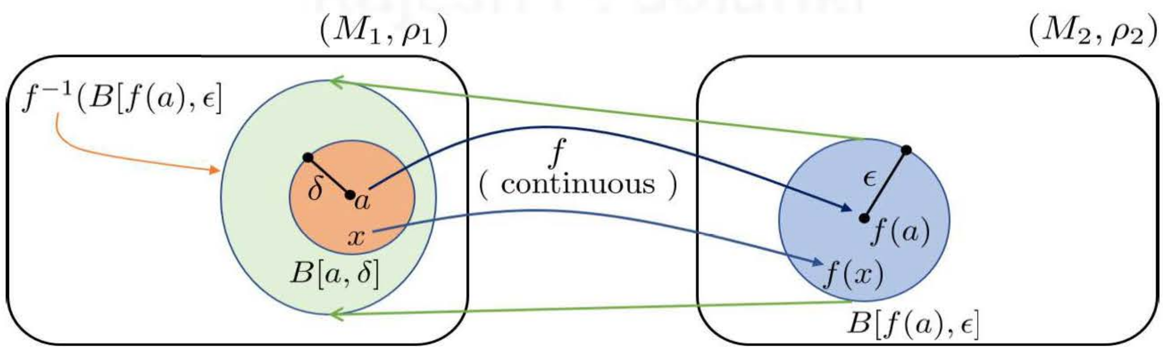
Let (M_1, ρ_1) and (M_2, ρ_2) be two metric spaces and $f : M_1 \to M_2$ be continous at $a \in M_1$.

$$\begin{aligned}
&\text{Now, } \; f \text{ is continuos at } a \in M_1 \text{ - - - (1)} \\
&\iff \lim_{x \to a} f(x) = f(a) \\
&\iff \text{ for a given } \epsilon > 0 \; \exists \; \text{ some } \delta > 0 \; \text{ such that} \\
&\qquad \rho_2(f(x), f(a)) < \epsilon \quad \text{whenever} \quad \rho_1(x, a) < \delta \text{ - - - (2)} \\
&\iff f(x) \in B[f(a), \epsilon] \quad \text{whenever} \quad x \in B[a, \delta] \\
&\iff f(B[a, \delta]) \subset B[f(a), \epsilon] \\
&\iff B[a, \delta] \subset f^{-1}(B[f(a), \epsilon]) \text{ - - - (3)}
\end{aligned}$$





$f(x) \in B[f(a), \epsilon] \quad$ whenever $\quad x \in B[a, \delta]$



$B[a, \delta] \subset f^{-1}(B[f(a), \epsilon])$

Thus we have $(a) \iff (b)$

Next we show that (b) and (c) are equivalent.

Let us assumet the following given at (3)

$$B[a,\delta] \subset f^{-1}(B[f(a),\epsilon]$$

As discussed above, it is equivalent to the following

$$f(x) \in B[f(a),\epsilon] \;\text{ whenever }\; x \in B[a,\delta] ----(4)$$

Now, suppose $\{x_n\}_{n=1}^{\infty}$ is any sequence of points in M_1 converging to a. i.e.

$$\lim_{n\to\infty} x_n = a$$

Therefore, for $\delta > 0$ there is some $N \in I$ such that,

$$\rho_1(x_n, a) < \delta, \quad \text{whenever} \quad n \geqslant N$$

$$\therefore\; x_n \in B[a,\delta], \quad \text{whenever} \quad n \geqslant N \text{ - - - } (5)$$

Taking $x = x_n$ in (4), it follows from (4) and (5) that,

$$f(x_n) \in B[f(a),\epsilon] \;\text{ whenever }\; n \geqslant N$$

Therefore

$$\lim_{n\to\infty} f(x_n) = f(a)$$

Thus,

$$\lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = f(a)$$

This shows $(b) \implies (c)$ - - - (6)

Conversely, suppose

$$\lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = f(a) \quad \text{- - - } (7)$$

If possible, suppose there exists some $\epsilon > 0$ such that the inverse image under f of $B[f(a);\epsilon]$ contains no open ball about a.

Therefore, for each positive integer n we have

$$B\left[a,\frac{1}{n}\right] \not\subset f^{-1}(B[f(a),\epsilon])$$

Therefore for each $n \in I$ there must be some $x_n \in M_1$ such that

$$x_n \in B\left[a,\frac{1}{n}\right] \quad \text{but} \quad x_n \notin f^{-1}(B[f(a),\epsilon])$$

It implies that,

$$x_n \in B\left[a, \frac{1}{n}\right] \quad \text{but} \quad f(x_n) \notin B[f(a), \epsilon]$$

Thus, get a sequence of points $\{x_n\}_{n=1}^{\infty}$ in M_1 such that,

$$\rho_1(x_n, a) < \frac{1}{n} \quad \text{but} \quad \rho_2(f(x_n), f(a)) \geqslant \epsilon$$

. Since $\lim_{n\to\infty} \frac{1}{n} = 0$ it is clear that

$$\lim_{n\to\infty} x_n = a$$

But $\rho_2(f(x_n) - f(a)) \geqslant \epsilon$ implies that

$$\lim_{n\to\infty} f(x_n) \neq f(a)$$

This is a contradiction to our assumption. So, our supposition is wrong.

Hence, for a given $\epsilon > 0$ there exists some $\delta > 0$ such that

$$\rho_2(f(x), f(a)) < \epsilon \quad \text{whenever} \quad \rho_1(x, a) < \delta$$

This shows $(c) \implies (b)$ - - - (7)

From (6) and (7) it follows that $(b) \iff (c)$

Hence

$$(a) \iff (b) \iff (c)$$

39. Let (M_1, ρ_1), (M_2, ρ_2) and (M_3, ρ_3) be metric spaces and let $f : M_1 \longrightarrow M_2$ and $g : M_2 \longrightarrow M_3$. If f is continuous at $a \in M_1$ and g is continuous at $f(a) \in M_2$, then prove that $g \circ f$ is continuous at a.

Proof:

As $f : M_1 \to M_2$ and $g : M_2 \to M_3$ the composite function $g \circ f$ is defined by

$$(g \circ f)(x) = g(f(x)), \ \forall x \in M_1$$

Now, suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence of points in M_1 converging to $a \in M_1$.
Therefore,

$$\lim_{n\to\infty} x_n = a$$

Also, f is continuous at a. Therefore,

$$\lim_{n\to\infty} f(x_n) = f(a)$$

Since, g is continuous at $f(a) \in M_2$, we must have,

$$\lim_{n\to\infty} g(f(x_n)) = g(f(a))$$

This implies that,

$$\lim_{n\to\infty}(g\circ f)(x_n)=(g\circ f)(a)$$

Hence, $g\circ f$ is continuous at a

40. **Let M be a metric space and let f and g be a real valued functions which are continuous at $a\in M$. Then prove that $f+g$, $f-g$. $f.g$ are also continuous at a. Furthermore, if $g(a)\neq 0$ then $\frac{f}{g}$ is also continuous at a.**

Proof:
Let (M,ρ) be a metric space. If $f:M\to R^1$ and $g:M\to R^1$ are continuous at $a\in M$ then

$$\lim_{x\to a}f(x)=f(a) \quad \text{and} \quad \lim_{x\to a}g(x)=g(a)$$

Now,

$$\begin{aligned}\lim_{x\to a}(f+g)(x)&=\lim_{x\to a}(f(x)+g(x))\\&=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)\\&=f(a)+g(a)\\&=(f+g)(a)\end{aligned}$$

As $\lim_{x\to a}(f+g)(x)=(f+g)(a)$ function $f+g$ is continuous at $a\in M$.

41. **Prove that every function from R_d into a metric space is continuous on R_d.**

Proof:
We have the discrete metric space R_d with discrete metric defined by $d:R\times R$ where

$$d(x,y)=\begin{cases}1\ , & \text{if } x\neq y\\ 0\ , & \text{if } x=y\end{cases}$$

So, for any $a\in R$ we get

$$\begin{aligned}B[a,1]&=\{x\in R/d(a,x)<1\}\\&=\{x\in R/d(a,x)=0\}\\B[a,1]&=\{a\}\end{aligned}$$

Let, $f:R_d\to M$ be a function definded on R_d into a metric space M. Now, for any $a\in R$ and $\epsilon>0$ we have

$$f(a)\in B[f(a),\epsilon]$$
$$\therefore\ a\in f^{-1}(B[f(a),\epsilon])$$
$$\therefore\ \{a\}\subset f^{-1}(B[f(a),\epsilon])$$
$$\therefore\ B[a,1]\subset f^{-1}(B[f(a),\epsilon])$$

This implies that, f is continuous at every $a \in R$.

Hence, every function defined on R_d is continuous.

42. Let $f : R^2 \longrightarrow R$ be a function defined by $f(x, y) = x,\ \forall x, y \in R^2$. Show that f is continuous on R^2.

Proof:
Here, $f : R^2 \to R$ is a defined by $f(x, y) = x,\ \forall x, y \in R^2$.

Therefore, for any $(a, b) \in R^2$ we have,

$$f(a, b) = a$$

Therefore,

$$|f(x, y) - f(a, b)| = |x - a|$$

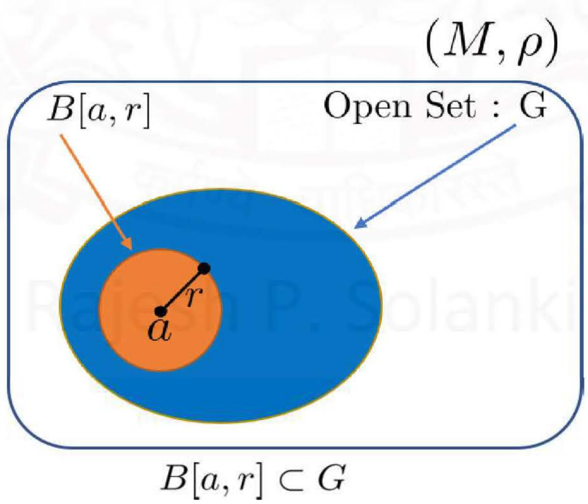
So, for any given $\epsilon > 0$ taking $\delta = \epsilon$ we get

$$|f(x, y) - f(a, b)| < \epsilon, \quad \text{whenwver} \quad |x - a| < \delta$$

Therefore, f is continuos at every (a, b). Hence f continuous on R^2.

43. Open Set

Open Set:
A subset of G of a metric space (M, ρ) is said to be open in the metric space if for every $x \in G$



there is some $r > 0$ such that

$$B[x, r] \subset G$$

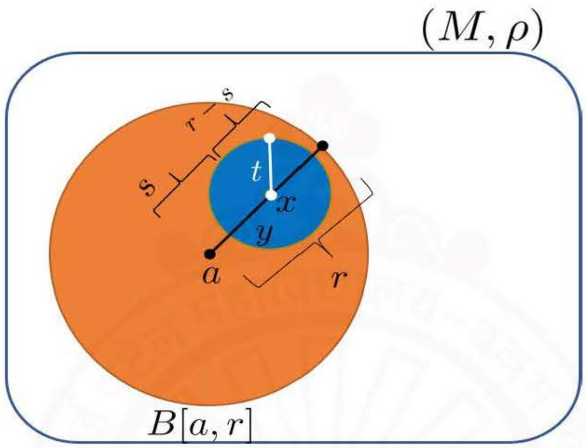
44. Prove that if (M, ρ) is a metric space then any open ball in M is an open set.

Let (M, ρ) be a metric space.
For any $a \in M$ and $r > 0$ consider the open ball $B[a, r]$.
Let $x \in B[a, r]$. Suppose $\rho(a, x) = s$. Clearly $s < r$.
As $r - s > 0$, we choose some positive number t such that

$$t < r - s$$

Consider the open ball $B[x, t]$.

Let $y \in B[x, t]$. Now,



$B[x, t] \subset B[a, r]$

$$\begin{aligned} \rho(a, y) &\leqslant \rho(a, x) + \rho(x, y) \\ &< s + t \\ &< s + (r - s) \\ &< r \end{aligned}$$

Since $\rho(a, y) < r$ we have $y \in B[a, r]$
Therefore $B[x, t] \subset B[a, r]$
As x is any point in $B[a, r]$, we can say that $B[a, r]$ contains an open ball for each of its members. Hence, $B[a, r]$ is an open set.

45. **For the discrete metric $\mathbb{R}_d$, find**
(1) $B[a; 2]$ **(2)** $B[a; 1/2]$ **(3)** $B[a; 1]$ **(4)** $B[a; -1/6]$**.**

Answer:
For the discrete metric $\mathbb{R}_d$
(1) $B[a; 2] = \{x \in R/d(x, a) < 2\} = \{x \in R \backslash d(x, a) = 1 \text{ or } d(x, a) = 0\} = R$

(2) $B[a; \frac{1}{2}] = \{x \in R/d(x, a) < \frac{1}{2}\} = \{x \in R/d(x, a) = 0\} = \{a\}$

(3) $B[a; 1] = \{x \in R/d(x, a) < 1\} = \{x \in R \backslash d(x, a) = 0\} = \{a\}$

(4) The open ball does not exist as its radius must be postive.

46. **Let $M = [0,1]$ with usual metric (absolute metric). Find (2) $B[\frac{1}{4}; \frac{1}{2}]$ (3) $B[\frac{1}{7}; 30]$ (4) $B[\frac{1}{7}; \frac{1}{7}]$.**

Answer:
(1)

$$B\left[\frac{1}{4}, 1\right] = \left(\frac{1}{4} - 1, \frac{1}{4} + 1\right) \cap [0,1]$$
$$= \left(-\frac{3}{4}, \frac{5}{4}\right) \cap [0,1]$$
$$= [0,1]$$

(2)

$$B\left[\frac{1}{4}, \frac{1}{2}\right] = \left(\frac{1}{4} - \frac{1}{2}, \frac{1}{4} + \frac{1}{2}\right) \cap [0,1]$$
$$= \left(-\frac{1}{4}, \frac{3}{4}\right) \cap [0,1]$$
$$= \left[0, , \frac{3}{4}\right)$$

(3) $B\left[\frac{1}{7}, 30\right] = \left(\frac{1}{7} - 30, \frac{1}{7} + 30\right) \cap [0,1] = [0,1]$

(4)

$$B\left[\frac{1}{7}, \frac{1}{7}\right] = \left(\frac{1}{7} - \frac{1}{7}, \frac{1}{7} + \frac{1}{7}\right) \cap [0,1]$$
$$= \left(0, \frac{2}{7}\right) \cap [0,1]$$
$$= \left(0, \frac{2}{7}\right)$$

47. **Prove that in any metric space (M, ρ), both M and ϕ are open sets.**

Answer:
In any metric space (M, ρ), For every $x \in M$ and $r > 0$ we have

$$B[x, r] \subset M$$

$\therefore$ the set M is a neighbourhood of all its points.
$\therefore$ M is an open set.

Moreover, there is no point in ϕ whose neighbourhood it cannot be.
$\therefore$ ϕ is also an open set.

48. **Let $\mathcal{F}$ be any non-empty family of open subsets of a metric space M. Then prove that $\bigcup_{G\in\mathcal{F}} G$ is also an open subset of M.**

Proof:
Here, $\mathcal{F}$ is a family of open subsets in a metric space M.
Let

$$H = \bigcup_{G\in\mathcal{F}} G$$

If $x \in H$ then there is some $G \in \mathcal{F}$ such that

$$x \in G$$

As G is member of $\mathcal{F}$ it is open in M. So there is some $r > 0$ such that

$$B[x,r] \subset G$$

$$\therefore\ B[x,r] \subset \bigcup_{G\in\mathcal{F}} G$$

$$\therefore B[x,r] \subset H$$

Therefore, if $x \in H$ then there is some open ball $B[x,r]$ such that $B[x,r] \subset H$.

Hence, H is an open set in M.

49. **Prove that every subset of $\mathbb{R}_d$ is open.**

Proof:
We have the discrete metric space R_d with discrete metric defined by $d : R \times R \to [0,\infty)$ where

$$d(x,y) = \begin{cases} 1\ , & \text{if } x \neq y \\ 0\ , & \text{if } x = y \end{cases}$$

So, for any $a \in R$ we get

$$\begin{aligned} B[a,1] &= \{x \in R / d(a,x) < 1\} \\ &= \{x \in R / d(a,x) = 0\} \\ B[a,1] &= \{a\} \end{aligned}$$

As every singletone subset in a discrete metric space R_d is an open ball, it is open in R_d.

Now, consider any subset A of R_d.

We can express A as a union of the singletone sets of its own elements as follows.

$$A = \bigcup_{x\in A} \{x\}$$

As each $\{x\}$ is an open set in R_d, we can say that A is a union of open subsets in R_d. Hence, A is an open set.

50. Is the intersection of an infinite number of open sets open? Justify!

Answer:
No.
Consider the infinite collection $\left\{\left(-\frac{1}{n}, \frac{1}{n}\right) / \text{ n is a positive integer } \right\}$ of open sets in R^1
Here,

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

Which is not open in R^1.

51. If G_1 and G_2 are open subsets of the metric space M, then prove that $G_1 \cap G_2$ is also open in M.

Proof:
Here, G_1 and G_2 are open subsets in a metric space M.

If $x \in G_1 \cap G_2$ then $x \in G_1$ and $x \in G_2$.

Therefore, there exist some $r_1 > 0$ and $r_2 > 0$ such that

$$B[x, r_1] \subset G_1 \quad \text{and} \quad B[x, r_2] \subset G_2 \text{ - - - } (1)$$

For $r = min\{r_1, r_2\}$ we have $r \leqslant r_1$ and $r \leqslant r_2$,
Therefore,

$$B[x, r] \subset B[x, r_1] \quad \text{and} \quad B[x, r] \subset B[x, r_2]$$

Therefore, from (1) it follows that,

$$B[x, r] \subset G_1 \quad \text{and} \quad B[x, r] \subset G_2$$

$$\therefore \quad B[x, r] \subset (G_1 \cap G_2)$$

Therefore, if $x \in G_1 \cap G_2$ then there is some $B[x, r]$ such that $B[x, r] \subset (G_1 \cap G_2)$

Hence, $G_1 \cap G_2$ is also open in M.

52. Prove that Every open subset G of $\mathbb{R}$ can be written $G = \bigcup I_n$, where $I_1, I_2, I_3, \ldots$ are a finite number or a countable number of open intervals which are mutually disjoint.

Proof:

Let G be an open subset of R^1. If $x \in G$ then there is an open interval I such that $x \in I \subset G$ Let I_x denote the largest such open interval containg x. Clearly,

$$G = \bigcup_{x \in G} I_x$$

Now, if $x \in G$ and $y \in G$ then for the largest open intervals I_x and I_y containg x and y respectively we must have

$$\text{either } I_x = I_y \quad \text{or} \quad I_x \cap I_y = \emptyset$$

For if $I_x \neq I_y$ and $I_x \cap I_y \neq \emptyset$ then $I_x \cup I_y$ is an open interval containing x. But then $I_x \cup I_y$ is be an open interval containing x which is larger than I_x.

This cotradicts our assumption that I_x is the largest open interval containg x.

Finally, each I_x contains a rational number. As disjoint open intervals cannot contain same rationals it is possible to assosiate each interval with a unique rational. Also since there are only countably many rationals, the disjoint open intervals I_x are finite or countable in number.

Thus, G is a union of finite number or a countable number of open intervals which are mutually disjoint.

53. Let (M_1, ρ_1) and (M_2, ρ_2) be two metric spaces and let $f : M_1 \to M_2$. Then prove that f is continuous on M_1 if and only if $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

<u>Proof:</u>

First suppose, $f : M_1 \to M_2$ is continuous on M_1. Consider an open subset G of M_2.

If $x \in f^{-1}(G)$ then $f(x) \in G$. As $f(x) \in G$ and G is an open subset of M_2, there is some open ball $B[f(x), s]$ such that

$$B[f(x), s] \subset G$$

Since, f is continuous at x there is some $B[x, r]$ such that

$$B[x, r] \subset f^{-1}(B[f(x), s]) \text{ - - - (1)}$$

But

$$B[f(x), s] \subset G \Rightarrow f^{-1}(B[f(x), s]) \subset f^{-1}(G) \text{ - - - (2)}$$

From, (1) and (2) it follows that,

$$B[x, r] \subset f^{-1}(G)$$

Therefore, if $x \in f^{-1}(G)$ there is some $B[x, r]$ such that $B[x, r] \subset f^{-1}(G)$

Hence, $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

Conversely, suppose $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

For any $a \in M_1$ we have $f(a) \in M_2$.

For $\epsilon > 0$ consider the open ball $B[f(a), \epsilon]$. As the open ball $B[f(a), \epsilon]$ is an open set in M_2 by our assumption $f^{-1}(B[f(a), \epsilon])$ is open in M_1.

Therefore, as $a \in f^{-1}(B[f(a), \epsilon])$ there is some $B[a, r]$ such that

$$B[a, r] \subset f^{-1}(B[f(a), \epsilon])$$

Thus, for every $a \in M_1$ we can always find some $B[a, r]$ such that $B[a, r] \subset f^{-1}(B[f(a), \epsilon])$.

Therefore, f is continuous at every $a \in M_1$.

Hence, f is continuous on M_1.

54. Prove that every constant function $f : R \to R$ is also continuous.

Proof:
Let, $f : R \to R$ be a constant function defined by $f(x) = k$, a constant, $\forall\, x \in R$
Therefore, for any $x, a \in R$,

$$|f(x) - f(a)| = |k - k| = 0$$

Therefore, For any $\epsilon > 0$, we always have

$$|f(x) - f(a)| < \epsilon$$

. Therefore, For any $\delta > 0$.

$$|f(x) - f(a)| < \epsilon \quad \text{whenever} \quad |x - a| < \delta$$

.
Therefore, f is continuous at every $a \in R$. Hence, f is continuous on R.

55. Limit Point

Limit Point:
Let (M, ρ) be a metric space and $E \subset M$. A point $x \in M$ is said to be a limit point of E if there is a sequence $\{x_n\}_{n=1}^{\infty}$ of points in E which converges to x. i.e.

$$\lim_{n \to \infty} x_n = x$$

56. Closure of a set.

Closure of a set:
Let (M, ρ) be a metric space and $E \subset M$. The set of all the limit points of E is called the closure of E and it is generally denoted by $\overline{E}$.

57. If E is any subset of the metric space M, then prove that $E \subset \overline{E}$.

Proof:
Let (M, ρ) be a metric space and $E \subset M$. Let x be any point in E.

As the sequence

$$x, x, x, \ldots$$

of points in E converges to x, point x is a limit point of E.

Therefore, every point of E is a limit point of E.

Hence,

$$E \subset \overline{E}$$

58. Closed set

Closed Set
A subset of a metric space is said to be closed if it contains all its limit points. In other words, if M is a metric space and $E \subset M$ then E is closed subset of M if

$$\overline{E} \subset E$$

Note:
If M is a metric space and $E \subset M$ then we always have $E \subset \overline{E}$. So in case E is a closed subset of M then we have

$$E \subset \overline{E} \quad \text{and } \overline{E} \subset E$$

Hence, E is closed iff $\overline{E} = E$

59. Let E be a subset of a metric space M. Then prove that a point $x \in M$ is a limit point of E *iff* every open ball $B[x; r]$ about x contains at least one point of E.

Proof:
Let E be a subset of a metric space (M, ρ).

First suppose, x is a limit point of E.

Therefore, there is a sequence of points $\{x_n\}$ in E which converges to x. Therefore, for any given $r > 0$ there is some positive integer N such that

$$\rho(x_n, x) < r \quad \text{whenever} \quad n \geqslant N$$

This implies that

$$x_n \in B[x,r] \quad \text{whenever} \quad n \geqslant N$$

As each $x_n \in E$ we conclude that $B[x,r]$ contains atleast one point of E.

Conversely suppose for each $r > 0$, $B[x,r]$ contains atleast one point of E.

Let $x_1 \in E$ such that $x_1 \in B[x,1]$,
$x_2 \in E$ such that $x_2 \in B[x, \frac{1}{2}]$
$x_3 \in E$ such that $x_3 \in B[x, \frac{1}{3}]$
Continuing similarly for each positive integer n, we get $x_n \in E$ such that $x_n \in B[x, \frac{1}{n}]$
Thus, we get a sequence $\{x_n\}$ of points in E such that,

$$\rho(x_n, x) < \frac{1}{n}, \ \forall\, n$$

Since, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ it follows that, sequence $\{x_n\}$ of points in E converges to x.

Hence, x is a limit point of E. We get a sequence $\{x_n\}$ of points.

60. If E is any subset of the metric space M, Then show that $\overline{E}$ is closed.

Proof:
To prove that $\overline{E}$ is closed, we shall show that $\overline{E} = \overline{\overline{E}}$

Since $\overline{E} \subset \overline{\overline{E}}$, it remians to show that $\overline{\overline{E}} \subset \overline{E}$.

Consider any $x \in \overline{\overline{E}}$. Since $x \in \overline{\overline{E}}$, x is a limit point of $\overline{E}$.

Therefore, any open ball $B[x,r]$ contains a point $y \in \overline{E}$.
Let $s = \rho(x,y)$ and choose any positive number t with $t < r - s$.
Since $y \in \overline{E}$, y is a limit point of E.
Therefore, the open ball $B[y,t]$ contains a point $z \in E$.
Now,

$$\begin{aligned} \rho(x,z) &\leqslant \rho(x,y) + \rho(y,z) \\ &< s + t \\ &< s + (r-s) \qquad (\because t < r - s) \\ &= r \\ \therefore \qquad \rho(x,z) &< r \end{aligned}$$

Hence, $z \in B[x;r]$.
Thus, for each $x \in \overline{\overline{E}}$, every open ball $B[x;r]$ contains a point in E.
Therefore each $x \in \overline{\overline{E}}$ is a limit point of E.

$$\therefore\ x \in \overline{\overline{E}} \Rightarrow x \in \overline{E}$$

Therefore,

$$\overline{\overline{E}} \subset \overline{E}$$

Hence,

$$\overline{\overline{E}} = \overline{E}$$

Therefore, $\overline{E}$ is a closed set.

61. Prove that in any metric space (M, ρ), the set M and $\emptyset$ are closed sets.

Proof:
In any metric space (M, ρ), for each $x \in M$ every open ball $B[x, r]$ always contains a point of M, as $B[x, r] \subset M$. Therefore, every $x \in M$ is a limit point of M. Therefore $\overline{M} = M$. Hence M is closed in M.

Also, empty set $\emptyset$ has no limit points. Therefore it is closed.

62. If F_1 and F_2 are closed subsets of the metric space M, then prove that $F_1 \cup F_2$ is also closed.

Proof:
If $x \in \overline{F_1 \cup F_2}$ then x is a limit point of $F_1 \cup F_2$.

Therefore, there is a sequence $\{x_n\}$ of points $F_1 \cup F_2$ converging to x.

In that case, $\{x_n\}$ must have a subsequence consisting of all its points in F_1 only or a subsequence consisting of all its points in F_2 only, converging to x.

This implies that either $x \in \overline{F_1}$ or $x \in \overline{F_2}$.

Therefore,

$$x \in \overline{F_1} \cup \overline{F_2}$$

As F_1 and F_2 both are closed in M, we have, $\overline{F_1} = F_1$ and $\overline{F_2} = F_2$. Therefore,

$$x \in F_1 \cup F_2$$

. Hence ,

$$\overline{F_1 \cup F_2} \subset F_1 \cup F_2$$

.
Therefore, $F_1 \cup F_2$ is closed.

63. Is it true that arbitrary union of closed sets is also closed? Justify!

Answer:
No, It is not neccessary.

In the metric space, R^1 Consider the collection

$$\left\{\left[\frac{1}{n}, 3-\frac{1}{n}\right] / n \in I\right\}$$

Here, all the closed intervals are closed in R^1.

But

$$\bigcup_{n=1}^{\infty}\left[\frac{1}{n}, 3-\frac{1}{n}\right]=(0,3)$$

Which is an open set.

Thus an arbitrary union of closed sets is not necessarily closed.

64. **If $\mathcal{F}$ is any family of closed subsets of a metric space M, then prove that $\bigcap\limits_{F\in\mathcal{F}} F$ is also closed.**

Proof:
Let $\mathcal{F}$ be any family of closed subset of a metric space M.

If $x \in \overline{\bigcap\limits_{F\in\mathcal{F}} F}$ then x is a limit point of $\bigcap\limits_{F\in\mathcal{F}} F$

Therefore, any open ball $B[x,r]$ contains atleast one point, say y, of $\bigcap\limits_{F\in\mathcal{F}} F$

Therefore, $B[x,r]$ contains atleast a point y of each F in $\mathcal{F}$.

This implies that x is a limit point of each $F \in \mathcal{F}$ Therefore,

$$y \in \overline{F},\ \forall F \in \mathcal{F}$$

But $\overline{F} = F$ as each F is a closed subset of M.

Therefore,

$$y \in F,\ \forall F \in \mathcal{F}$$

Hence,

$$x \in \bigcap_{F\in\mathcal{F}} F$$

Thus, we have

$$\overline{\bigcap_{F\in\mathcal{F}} F} \subset \bigcap_{F\in\mathcal{F}} F$$

Hence, $\bigcap\limits_{F \in \mathcal{F}} F$ is a closed subset of M.

This proves that arbitrary intersection of closed set is closed.

65. Prove that a subset G of the metric space M is open *iff* compliment of G is closed.

Proof:

Let (M, ρ) be a metric space.

First suppose G is an open subset of M.

Now, to show that $F = M - G$ is a closed subset of M it is sufficient to show that F contains all its limit points.

For this we shall show that G cannot contain any limit point of F.

Consider $x \in G$. As G is open in M there is an open ball $B[x, r]$ such that

$$B[x, r] \subset G$$

Therefore,

$$B[x, r] \cap F = \emptyset$$

Thus, there is an open ball $B[x, r]$ which does not contain any point of F. This implies that x cannot be a limit point of F.

Hence, F must contain all its limit points. Therefore, $F = M - G$ is closed whenever G is open.

Next, suppose F is a closed subset of M. Therefore it must contain all its limit points.

Let $G = M - F$. So, if $x \in G$ then $x \notin F$.

Therefore, x cannot be a limit point of F. But then there must be some open ball centered at x, say $B[x, r]$, such that $B[x, r]$ does not contain any point in F. Therefore,

$$B[x, r] \subset G$$

As x is any point in G, we conclude that $G = M - F$ is open whenever F is closed.

66. Let (M_1, ρ_1) and (M_2, ρ_2) be metric spaces and let $f : M_1 \to M_2$. Then prove that f is continuous on M_1 if and only if $f^{-1}(F)$ is closed subset of M_1 whenever F is a closed subset of M_2.

Proof:

First, suppose that $f : M_1 \to M_2$ is continuous on M_1

Now, if F is a closed subset of M_2 then its complement $F' = M_2 - F$ is open in M_2.

Since, f is continuous on M_1, we must have $f^{-1}(F')$ open in M_1.
Also,

$$\begin{aligned} & F \cup F' = M_2 \text{ and } F \cap F' = \emptyset \\ \therefore \quad & f^{-1}(F \cup F') = f^{-1}(M_2) \\ \therefore \quad & f^{-1}(F) \cup f^{-1}(F') = M_1 \\ \therefore \quad & f^{-1}(F) = M_1 - f^{-1}(F') \end{aligned}$$

As $f^{-1}(F')$ is an open subset of M_1, its complement $f^{-1}(F)$ is closed in M_1.

Conversely, suppose $f^{-1}(F)$ is closed subset of M_1 whenever F is a closed subset of M_2.

Let G be an open subset of M_2. Therefore $F = M_2 - G$ is closed in M_2.

Also,

$$\begin{aligned} & F \cup G = M_2 \text{ and } F \cap G = \emptyset \\ \therefore \quad & f^{-1}(F \cup G) = f^{-1}(M_2) \\ \therefore \quad & f^{-1}(F) \cup f^{-1}(G) = M_1 \\ \therefore \quad & f^{-1}(G) = M_1 - f^{-1}(F) \end{aligned}$$

As $f^{-1}(F)$ is a closed subset of M_1, its complement $f^{-1}(G)$ is open in M_1.

Thus, $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

Hence, f is continuous on M_1.

67. Homeomorphism

Homeomorphism:
Let (M_1, ρ_1) and (M_2, ρ_2) be two metric spaces. A function f is said to be a Homeomorphism from M_1 onto M_2 if $f : M_1 \to M_2$ is one-one and M_1 onto M_2 and possess all the following equivalent propertires.
(1) Both f and f^{-1} are continuous.

(2) A set $G \subset M_1$ is open *iff* its image $f(G) \subset M_2$ is open.

(3) A set $F \subset M_1$ is closed *iff* its image $f(F) \subset M_2$ is closed.

68. Dense Set

Dense Set:
Let (M, ρ) be a metric space. A subset A of M is said to be dense in A if $\bar{(A)} = M$.

69. Show that no proper subset of $\mathbb{R}_d$ is dense.

Answer:
In R_d, the discrete metric space on R, we have the discrete metric defined by,

$$d(x,y) = \begin{cases} 1\ , & \text{if} \quad x \neq y \\ 0\ , & \text{if} \quad x = y \end{cases}$$

Now, let A be a non-empty proper subset of R_d. There are some members of R which are not in A.

If $x \in R - A$ and $y \in A$ then clearly $x \neq y$. Therefore,

$$d(x,y) = 1$$

As, $B[x,1] = \{y \in R/d(x,y) < 1\}$, it cannot contain a point of A.

Therefore, any point x in $R - A$ cannot be a limit point of A.

Therefore,

$$\overline{A} \neq R$$

in R_d.

Hence, no proper subset of R_d is dense.

70. Give an example of a set E such that E and its compliment are dense in R.

Answer:
Let $E = \{x \in R/x \text{ is a rational number }\}$. Therefore, the complement of E is

$$E' = R - E$$

,which is the set of all irrational numbers.

As, every open interval containing any real number contains infintely many rationals as well as irrationals, all the real numbers are limit points of E and E' both.

Therefore,

$$\overline{E} = R \quad \text{and} \quad \overline{E'} = R$$

Hence, E and its complement E' both are dense in R.

71. Prove that any finite subset of a metric space is closed.

Proof:
Let A be a finite subset of a metric space (M, ρ). Suppose $A = \{x_1, x_2, \ldots, x_n\}$

If $x \in M - A$ then define,

$$r = min\{\rho(x, x_1), \rho(x, x_2), \ldots, \rho(x, x_n)\}$$

Clearly,

$$r \leqslant \rho(x, x_i), \ \forall i = 1, 2, \ldots, n$$

As $B[x, r] = \{y \in M / \ \rho(x, y) < r, \}$ it cannot contain any point of A.

Therefore, no point of $M - A$ can be a limit point of A.

Therefore,

$$\overline{A} = A$$

Hence, A is closed.

Thus, every finite subset of a metric space is closed.