T.Y.B.Sc. : Semester - V

US05CMTH22(T)

Theory Of Real Functions

[Syllabus effective from June , 2020]

Study Material Prepared by :
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Unit:1
Limits , Continuous Function , Functions Continuous on Closed and Bounded Intervals , Uniform Continuity ,Derivability of a Function , Properties of Derivable Functions.

Unit:2
Increasing and Decreasing Functions , Darboux Theorem , Rolle's Theorem , Lagrange's and Cauchy's Mean Value Theorems , Taylor's Theorem with Lagrange's Form of Remainder and Cauchy's Form of Remainder , Maclaurin's Theorem , Generalized Mean Value Theorem , Taylor's and Maclaurin's Series Expansions of Exponential and Trigonometric Functions , $log(1+x)$ and $(1+x)^n$

Unit:3
Functions of Several Variables: Explicit and Implicit Functions , Continuity , Partial Derivatives , Differentiability , Partial Derivatives of higher order , Differentils of Higher Order,Functions of Function

Unit:4
Change of Variables, Taylor's Theorem and Maclaurin's Theorem for Function of Two Variables ; Extreme Values of Functions of Two Variables.

Recommended Textbooks :
1. Principals of Real Analysis
   Author : S.C.Malik
   Publisher : New Age International, New Delhi
   Edition : 3rd Ed.
   Edition : Ch. 15,6,11 (Except 11.11).

Recommended Reference Books :

1. Elementary Analysis : The Theory of Calculus
   Author : K.A.Rose
   Edition : 2009
   Publisher : Springer (SIE), Indian reprint

2. Introduction to Real Analysis
   Author : R.G.Bartle,D.R.Sherbert
   Edition : Third Edition
   Publisher : Wiley India Pvt.Ltd.New Delhi

3. A Course in Calculus and Real Analysis
   Author : S.R.Ghorpade and B.V.Limaye
   Edition : 2006
   Publisher : Springer

4. Introduction to Analysis
   Author : A.Mattuck
   Edition : 1999
   Publisher : Prentice Hall

5. Mathematical Analysis
   Author : S.C.Malik and Savita Arora
   Edition : Second Edition, 2000
   Publisher : New Age International Pvt. Ltd., New Delhi

6. Real Analysis
   Author : Dipak Chatterjee
   Edition :
   Publisher : Prentice -Hall India Pvt. Ltd.New Delhi

US05CMTH22(T)- UNIT : I

1. Limit of a function

Limit of a function :
Let f be a function whose domain contains a neighbourhood of a real number c and l be a fixed real number. If for each $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - l| < \epsilon \text{ whenever } |x - c| < \delta$$

then l is said to be the limit of f as x tends to c and it is written as

$$\lim_{x \to c} f(x) = l$$

Remark:
We know the following equivalence

$$|x - c| < \delta \Leftrightarrow x \in (c - \delta, c + \delta) \Leftrightarrow c - \delta < x < +\delta$$

similarly,

$$|f(x) - l| < \epsilon \Leftrightarrow f(x) \in (l - \epsilon, l + \epsilon) \Leftrightarrow l - \epsilon < x < l + \epsilon$$

Hence, the $\epsilon - \delta$ condition in the definition of limit can be expressed by replacing $|f(x) - l|$ and $|x - c| < \delta$ by their equivalent forms.

So, any of the following conditions can replace the condition used in the definition.

$$f(x) \in (l - \epsilon, l + \epsilon) \text{ whenever } x \in (c - \delta, c + \delta)$$

and

$$|f(x) - l| < \epsilon \text{ whenever } c - \delta < x < +\delta$$

2. Left Hand Limit

Left Hand Limit of a function :
Let f be a function and c be real number such that domain of f contains some interval (a, c). A real number l is said to be the limit of f as x tends to c from left, if for each $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - l| < \epsilon \text{ whenever } x \in (c - \delta, c)$$

In symbols it is written as follows,

$$\lim_{x \to c-} f(x) = l$$

3. Right Hand Limit

Right Hand Limit of a function :
Let f be a function and c be real number such that domain of f contains some interval (c, a). A real number l is said to be the limit of f as x tends to c from right, if for each $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - l| < \epsilon \text{ whenever } x \in (c, c+\delta)$$

In symbols it is written as follows,

$$\lim_{x \to c+} f(x) = l$$

4. An important result

An important result

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f(x)} = \mathbf{l} \iff \lim_{\mathbf{x} \to \mathbf{a}-} \mathbf{f(x)} = \mathbf{l} = \lim_{\mathbf{x} \to \mathbf{a}+} \mathbf{f(x)}$$

5. Prove that limit of a function is unique, if it exists.

Proof
Suppose $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = l_1$ and $\lim_{x \to a} f(x) = l_2$

Therefore, for any $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - l_1| < \frac{\epsilon}{2} \text{ whenever } 0 < |x - a| < \delta$$

and

$$|f(x) - l_2| < \frac{\epsilon}{2} \text{ whenever } 0 < |x - a| < \delta$$

Now,

$$\begin{aligned}
|l_1 - l_2| &= |l_1 - f(x) + f(x) - l_2| \\
&= |(l_1 - f(x)) + (f(x) - l_2)| \\
&\leqslant |l_1 - f(x)| + |f(x) - l_2| \\
&= |f(x) - l_1| + |f(x) - l_2| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
|l_1 - l_2| &< \epsilon
\end{aligned}$$

As ϵ is any positive number and $|l_1 - l_2| < \epsilon$ it follows that the non-negative number $|l_1 - l_2|$ is less than every positive number.
This implies that $|l_1 - l_2| = 0$, hence

$$l_1 = l_2$$

.

Which proves that there cannot be more than one limits.
Hence, $\lim_{x \to a} f(x)$ is unique if it exists.

6. **Let f and g be two functions defined on some neighbourhood of a such that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$. Prove the following.**

[1] $\lim_{x \to a} [f(x) + g(x)] = l + m$

Proof
Here,

$$\lim_{x \to a} f(x) = l$$

Therefore, for each $\epsilon > 0$ there exists some $\delta_1 > 0$ such that

$$|f(x) - l| < \frac{\epsilon}{2} \text{ whenever } 0 < |x - a| < \delta_1$$

Also as $\lim_{x \to a} g(x) = m$, for the same ϵ there exists some $\delta_2 > 0$ such that

$$|g(x) - m| < \frac{\epsilon}{2} \text{ whenever } 0 < |x - a| < \delta_2$$

If we take, $\delta = min\{\delta_1, \delta_2\}$ then $\delta \leqslant \delta_1$ and $\delta \leqslant \delta_2$.
Hence,

$$|f(x) - l| < \frac{\epsilon}{2} \text{ whenever } 0 < |x - a| < \delta$$

and

$$|g(x) - m| < \frac{\epsilon}{2} \text{ whenever } 0 < |x - a| < \delta$$

Therefore for $0 < |x - a| < \delta$,

$$\begin{aligned} |(f(x) + g(x)) - (l + m)| &= |(f(x) - l) + (g(x) - m)| \\ &\leqslant |f(x) - l| + |g(x) - m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ \therefore |(f(x) + g(x)) - (l + m)| &< \epsilon \end{aligned}$$

Since

$$|(f(x) + g(x)) - (l + m)| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

we conclude that,

$$\lim_{x \to a} [f(x) + g(x)] = l + m$$

[2] $\lim_{x \to a} [f(x) - g(x)] = l - m$

Proof

Here,

$$\lim_{x \to a} f(x) = l$$

Therefore, for each $\epsilon > 0$ there exists some $\delta_1 > 0$ such that

$$|f(x) - l| < \frac{\epsilon}{2} \text{ whenever } 0 < |x - a| < \delta_1$$

Also as $\lim_{x \to a} g(x) = m$, for the same ϵ there exists some $\delta_2 > 0$ such that

$$|g(x) - m| < \frac{\epsilon}{2} \text{ whenever } 0 < |x - a| < \delta_2$$

If we take, $\delta = min\{\delta_1, \delta_2\}$ then $\delta \leqslant \delta_1$ and $\delta \leqslant \delta_2$.
Hence,

$$|f(x) - l| < \frac{\epsilon}{2} \text{ whenever } 0 < |x - a| < \delta$$

and

$$|g(x) - m| < \frac{\epsilon}{2} \text{ whenever } 0 < |x - a| < \delta$$

Therefore for $0 < |x - a| < \delta$,

$$\begin{aligned}
|(f(x) - g(x)) - (l - m)| &= |(f(x) - l) + (m - g(x))| \\
&\leqslant |f(x) - l| + |m - g(x)| \\
&= |f(x) - l| + |g(x) - m| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}
\end{aligned}$$

$$\therefore |(f(x) - g(x)) - (l - m)| < \epsilon$$

Since

$$|(f(x) - g(x)) - (l - m)| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

we conclude that,

$$\lim_{x \to a} [f(x) - g(x)] = l - m$$

[3] $\lim_{x \to a} [f(x)g(x)] = lm$

Proof

We have,

$$|f(x)g(x) - lm| = |f(x)g(x) - g(x)l + g(x)l - lm|$$

$$= \left|g(x)\big(f(x) - l\big) + l\big(g(x) - m\big)\right|$$

$$\leqslant \left|g(x)\big(f(x) - l\big)\right| + \left|l\big(g(x) - m\big)\right|$$

$$\leqslant |g(x)|.|f(x) - l| + |l|.|g(x) - m|$$

$$\therefore |f(x)g(x) - lm| \leqslant |g(x)|.|f(x) - l| + |l|.|g(x) - m| \ \text{- - -} \ (1)$$

As $\lim_{x \to a} g(x) = m$, for $\epsilon = 1$ there exists some $\delta_1 > 0$ such that

$$|g(x) - m| < 1 \ \text{ whenever } 0 < |x - a| < \delta_1$$

Now,

$$\begin{aligned} |g(x)| &= |g(x) - m + m| \\ &\leqslant |g(x) - m| + |m| \\ &\leqslant 1 + |m| \ \text{ when } 0 < |x - a| < \delta_1 \end{aligned}$$

Therefore, $|g(x)| \leqslant |m| + 1$ whenever $0 < |x - a| < \delta_1$

So for $0 < |x - a| < \delta_1$, from (1) we have,

$$|f(x)g(x) - lm| \leqslant (|m| + 1).|f(x) - l| + |l|.|g(x) - m| \ \text{- - -} \ (2)$$

Again considering the limits

$$\lim_{x \to a} f(x) = l \ \text{ and } \ \lim_{x \to a} g(x) = m$$

for each $\epsilon > 0$ there exists some $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$|f(x) - l| < \frac{\epsilon}{2(|m| + 1)} \ \text{ whenever } 0 < |x - a| < \delta_2$$

and

$$|g(x) - m| < \frac{\epsilon}{2(|l| + 1)} \ \text{ whenever } 0 < |x - a| < \delta_3$$

If we take, $\delta = min\{\delta_1, \delta_2, \delta_3\}$ then $\delta \leqslant \delta_1$, $\delta \leqslant \delta_2$ and $\delta \leqslant \delta_3$.
Hence,

$$|f(x) - l| < \frac{\epsilon}{2(|m| + 1)} \ \text{ whenever } 0 < |x - a| < \delta$$

and

$$|g(x) - m| < \frac{\epsilon}{2(|l| + 1)} \ \text{ whenever } 0 < |x - a| < \delta$$

Therefore, for $0 < |x - a| < \delta$ from (2) it follows that,

$$\begin{aligned} |f(x)g(x) - lm| &\leqslant (|m| + 1).\frac{\epsilon}{2(|m| + 1)} + |l|.\frac{\epsilon}{2(|l| + 1)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \end{aligned}$$

$$\therefore |f(x)g(x) - lm| < \epsilon$$

Since

$$|f(x)g(x) - lm| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

we conclude that,

$$\lim_{x \to a} f(x)g(x) = lm$$

[4] $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}$, **if** $m \neq 0$

Proof

We have,

$$\left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| = \left|\frac{mf(x) - lg(x)}{mg(x)}\right|$$

$$= \left|\frac{mf(x) - lm + lm - lg(x)}{mg(x)}\right|$$

$$= \frac{\big|m\big(f(x) - l\big) + l\big(m - g(x)\big)\big|}{|m||g(x)|}$$

$$\leqslant \frac{|m||f(x) - l|}{|m||g(x)|} + \frac{|l||g(x) - m|}{|m||g(x)|}$$

$$\therefore \left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| \leqslant \frac{1}{|g(x)|}.|f(x) - l| + \frac{|l|}{|m||g(x)|}|g(x) - m| \text{ --- (1)}$$

As $m \neq 0$ we have $|m| > 0$, hence $\frac{|m|}{2} > 0$

Since, $\lim_{x \to a} g(x) = m$ there exists some $\delta_1 > 0$ such that

$$|g(x) - m| < \frac{|m|}{2} \text{ whenever } 0 < |x - a| < \delta_1$$

Now,

$$\begin{aligned} |m| &= |m - g(x) + g(x)| \\ &\leqslant |g(x) - m| + |g(x)| \\ &\leqslant \frac{|m|}{2} + |g(x)| \end{aligned}$$

$$|m| - \frac{|m|}{2} <\leqslant |g(x)|$$

$$\frac{|m|}{2} \leqslant |g(x)|$$

Therefore, $\frac{1}{|g(x)|} \leqslant \frac{2}{|m|}$ whenever $0 < |x - a| < \delta_1$

So for $0 < |x - a| < \delta_1$, from (1) we have,

$$\left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| \leqslant \frac{2}{|m|}|f(x) - l| + \frac{2|l|}{|m|^2}|g(x) - m| \quad \text{- - - (2)}$$

Again we consider the limits

$$\lim_{x \to a} f(x) = l \quad \text{and} \quad \lim_{x \to a} g(x) = m$$

Therefore, for each $\epsilon > 0$ there exists some $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$|f(x) - l| < \frac{\epsilon|m|}{4} \quad \text{whenever } 0 < |x - a| < \delta_2$$

and

$$|g(x) - m| < \frac{\epsilon|m|^2}{4(|l| + 1)} \quad \text{whenever } 0 < |x - a| < \delta_3$$

If we take, $\delta = min\{\delta_1, \delta_2, \delta_3\}$ then $\delta \leqslant \delta_1$, $\delta \leqslant \delta_2$ and $\delta \leqslant \delta_3$.
Hence,

$$|f(x) - l| < \frac{\epsilon|m|}{4} \quad \text{whenever } 0 < |x - a| < \delta$$

and

$$|g(x) - m| < \frac{\epsilon|m|^2}{4(|l| + 1)} \quad \text{whenever } 0 < |x - a| < \delta$$

Therefore for $0 < |x - a| < \delta$ from (2) it follows that,

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| &< \frac{2}{|m|}\left(\frac{\epsilon|m|}{4}\right) + \frac{2|l|}{|m|^2}\left(\frac{\epsilon|m|^2}{4(|l| + 1)}\right) \\ &< \frac{\epsilon}{2} + \left(\frac{|l|}{|l| + 1}\right)\frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ \therefore \left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| &< \epsilon \end{aligned}$$

Since

$$\left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

we conclude that,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}$$

7. Continuity at an interior point

Continuity at an interior point:
A function f is said to be continuous at a point $c \in (a, b)$, if

$$\lim_{x \to c} f(x) = f(c)$$

In other words, the function f is continuous at $c \in (a, b)$ if for each $\epsilon > 0, \exists$ some $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon, \quad \text{whenever} \quad |x - c| < 0$$

8. Continuity from the left

Continuity from the left

A function f is said to be continuous from left at a point c if the limit $\lim\limits_{x \to c-} f(x) = f(c)$

9. Continuity from the right

Continuity from the right

A function f is said to be continuous from right at a point c if the limit $\lim\limits_{x \to c+} f(x) = f(c)$

10. Continuity at an end point

Continuity at an end point

A function f is said to be continuous at the end point a of a closed interval $[a, b]$ if it is right continuous at a, i.e.

$$\lim_{x \to a+} f(x) = f(a)$$

Also, f is said to be continuous at the end point b, if it is left continuous at b, i.e.

$$\lim_{x \to b-} f(x) = f(b)$$

11. Continuity in an interval

Continuity in an interval

A function f is said to be continuous in an interval if it is continuous at every point of the interval.

12. If f and g are two functions which are continuous at a then prove that $f + g$, $f - g$, fg and $\dfrac{f}{g}$ are also continuous at a

Here, f and g are two functions which are continuous at a. Therefore

$$\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a} g(x) = g(a)$$

Now we shall prove continuity of $f+g$ at c.

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\ &= \lim_{x\to a} f(x)+\lim_{x\to a} g(x) \\ &= f(a)+g(a) \\ &= (f+g)(a)\end{aligned}$$

As,

$$\lim_{x\to a}(f+g)(x)=(f+g)(a)$$

$f+g$ is continuous at a.

Similarly the continuity of $f-g, fg$ and $\dfrac{f}{g}$ can be proved.

13. **Evaluate :** $\lim_{x\to -1}\dfrac{(x+2)(3x-1)}{x^2+3x-2}$

$$\lim_{x\to -1}\frac{(x+2)(3x-1)}{x^2+3x-2}=\frac{(-1+2)(3(-1)-1)}{(-1)^2+3(-1)-2}=\frac{(1)(-4)}{-4}=1$$

14. **Evaluate :** $\lim_{x\to 0}\dfrac{\sqrt{4+x}-2}{x}$

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x} &= \lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}\times\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} \\ &= \lim_{x\to 0}\frac{4+x-4}{x(\sqrt{4+x}+2)} \\ &= \lim_{x\to 0}\frac{1}{\sqrt{4+x}+2} \\ &= \frac{1}{4}\end{aligned}$$

15. **Evaluate :** $\lim_{x\to 0}\dfrac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1}$

We shall evaluate left-hand and right hand limits separately.

$$\begin{aligned}x\to 0- &\Rightarrow \frac{1}{x}\to -\infty \\ &\Rightarrow e^{\frac{1}{x}}\to 0 \quad (\text{ because } e>1\ )\end{aligned}$$

Therefore, $\lim_{x\to 0-} \frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1} = \frac{0}{0+1} = 0$
Also,

$$\begin{aligned} x \to 0+ &\Rightarrow \frac{1}{x} \to +\infty \\ &\Rightarrow e^{\frac{1}{x}} \to +\infty \quad (\text{ because } e > 1 \text{ }) \\ &\Rightarrow e^{-\frac{1}{x}} \to 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{x\to 0+} \frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1} &= \lim_{x\to 0+} \frac{1}{1+e^{-\frac{1}{x}}} \\ &= \frac{1}{1+0} \\ &= 1 \end{aligned}$$

Since,

$$\lim_{x\to 0-} \frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1} \neq \lim_{x\to 0+} \frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1}$$

$\lim_{x\to 0} \frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1}$ does not exist.

16. Sequence.

Sequence

A function $f : N \longrightarrow R$, whose domain is the set of natural numbers and the range is subset of real numbers is called a sequence of real numbers. $f(1), f(2), \ldots$ are called the $1^{st}, 2^{nd}, \ldots$ terms of the sequence.

Generally, a sequence is represented as follows,

$$a_1, a_2, \ldots, a_n, \ldots$$

and instead of using the function notation it is denoted by

$$\{a_n\}$$

17. Limit of a Sequence.

Limit of a Sequence (Or Convergence of a Sequence)

Let $\{c_n\}$ be a sequence of real numbers. If for a fixed real number c, to every $\epsilon > 0$ there exists some positive integer m such that

$$|c_n - c| < \epsilon \quad \text{whenever} \quad n \geqslant m$$

then sequence $\{c_n\}$ is said to be *convergent* to c or equivalently c is said to be the limit of sequence $\{c_n\}$ and in symbols it is witten as

$$\lim_{n\to\infty} c_n = c$$

18. Show that a function $f : [a, b] \to \Re$ is continuous at point c of [a,b] iff

$$\lim_{n\to\infty} c_n = c \Longrightarrow \lim_{n\to\infty} f(c_n) = f(c)$$

Proof :

Suppose, a function f is continuous at a point c in an interval I.
Therefore, for any given $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \text{ whenever } |x - c| < \delta \quad \text{--- (1)}$$

Now, let $\{c_n\}$ be a sequence of points in I converging to c.
i.e.

$$\lim_{n\to\infty} c_n = c$$

Therefore, for $\delta > 0$ there exists some positive integer m such that

$$|c_n - c| < \delta, \quad \text{whenever } n \geqslant m \quad \text{--- (2)}$$

Taking $x = c_n$ in (1), from (1) and (2) it follows that,

$$|f(c_n) - f(c)| < \epsilon \text{ , whenever } n \geqslant m \quad \text{--- (3)}$$

Therefore,

$$\boxed{\lim_{n\to\infty} \mathbf{f(c_n) = f(c)}}$$

whenever $\lim_{n\to\infty} c_n = c$

Now, let us prove the converse by assuming

$$\lim_{n\to\infty} c_n = c \Longrightarrow \lim_{n\to\infty} f(c_n) = f(c)$$

If possible, suppose f is not continuous at c.

Therefore, there must be some $\epsilon > 0$ such that for any choice of $\delta > 0$ there is atleast one $x \in I$ such that

$$|f(x) - f(c)| \geqslant \epsilon \text{ when } |x - c| < \delta \quad \text{--- (4)}$$

Taking $\delta = 1$ in (1), we must have some $x = c_1 \in I$ such that

$$|f(c_1) - f(c)| \geqslant \epsilon \text{ when } |c_1 - c| < 1$$

Again, taking $\delta = \frac{1}{2}$ in (1), we must have some $x = c_2 \in I$ such that

$$|f(c_2) - f(c)| \geqslant \epsilon \text{ when } |c_2 - c| < \frac{1}{2}$$

Similarly, taking $\delta = \dfrac{1}{3}$ in (1), we must have some $x = c_3 \in I$ such that

$$|f(c_3) - f(c)| \geqslant \epsilon \text{ when } |c_3 - c| < \frac{1}{3}$$

Continuing in this manner by taking $\delta = \dfrac{1}{n}$ for each positive integer n, we shall get a sequence $\{c_n\}$ of points in I such that,

$$|f(c_n) - f(c)| \geqslant \epsilon \text{ when } |c_n - c| < \frac{1}{n}$$

As $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, clearly $\lim\limits_{n\to\infty} c_n = c$

But, on the other hand for each c_n we always have,

$$|f(c_n) - f(c)| \geqslant \epsilon$$

It follows that,

$$\lim_{n\to\infty} f(c_n) \neq f(c)$$

This contradicts our assumption. Therefore our suppossion that f is not continuous at c is wrong.
Hence, if

$$\lim_{n\to\infty} c_n = c \Longrightarrow \lim_{n\to\infty} f(c_n) = f(c)$$

then f is continuous at c.

19. Discontinuity

Discontinuity

A function f is said to be discontinuous at a point c if it is not continuous there at.

20. Removable Discontinuity

Removable Discontinuity:

If for a function f and $c \in R$ the limit $\lim\limits_{x\to c} f(x)$ exists but $\lim\limits_{x\to c} f(x)$ is not equal to $f(c)$, which may or may not exist, then f is said to have a removable discontinuity at c.

21. Discontinuity of first kind

Discontinuity of First Kind:

A function f is said to have a discontinuity of first kind, if both the limits

$$\lim_{x\to a-} f(x) \quad \text{and} \quad \lim_{x\to a+} f(x)$$

exist but are not equal.

22. Discontinuity of first kind from left.

Discontinuity of First kind from left:
A function f is said to have a discontinuity of first kind from left at $x = c$ if the limit

$$\lim_{x \to c-} f(x)$$

exists but is not equal to $f(c)$.

23. Discontinuity of first kind from right.

Discontinuity of First kind from right:
A function f is said to have a discontinuity of first kind from right at $x = c$ if the limit

$$\lim_{x \to c+} f(x)$$

exists but is not equal to $f(c)$

24. Discontinuity of second kind.

Discontinuity of Second kind:
A function f is said to have a discontinuity of second kind at $x = c$ if neither $\lim\limits_{x \to c-} f(x)$ nor $\lim\limits_{x \to c+} f(x)$ exists.

25. Discontinuity of second kind from left.

Discontinuity of Second kind from left:
A function f is said to have a discontinuity of second kind from left at $x = c$ if the limit

$$\lim_{x \to c-} f(x)$$

does not exist.

26. Discontinuity of second kind from right.

Discontinuity of Second kind from right:
A function f is said to be have a discontinuity of second kind from right at $x = c$ if the limit

$$\lim_{x \to c+} f(x)$$

does not exist.

27. Examine the following function for continuity at $x = 0$

$$f(x) = \begin{cases} \dfrac{xe^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}} & \textbf{when} \quad x \neq 0 \\ 0, & \textbf{when} \quad x = 0 \end{cases}$$

Let us evaluate the left-hand and right hand limits separately.
We have,

$$x \to 0- \Rightarrow \frac{1}{x} \to -\infty$$
$$\Rightarrow e^{\frac{1}{x}} \to 0 \quad (\text{ because } e > 1\ )$$

Therefore, $\lim\limits_{x\to 0-} \dfrac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} = \dfrac{0.0}{1+0} = 0$
Therefore,

$$\lim_{x\to 0-} f(x) = 0$$

Also,

$$x \to 0+ \Rightarrow \frac{1}{x} \to +\infty$$
$$\Rightarrow e^{\frac{1}{x}} \to +\infty \quad (\text{ because } e > 1\ )$$
$$\Rightarrow e^{-\frac{1}{x}} \to 0$$

Therefore,

$$\lim_{x\to 0+} \frac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} = \lim_{x\to 0+} \frac{x}{e^{-\frac{1}{x}}+1}$$
$$= \frac{0}{0+1}$$
$$= 0$$

Therefore,

$$\lim_{x\to 0+} f(x) = 0$$

Since,

$$\lim_{x\to 0-} \frac{xe^{\frac{1}{x}}}{e^{\frac{1}{x}}+1} = \lim_{x\to 0+} \frac{xe^{\frac{1}{x}}}{e^{\frac{1}{x}}+1} f(0) = 0$$
$$\lim_{x\to 0} f(x) = f(0)$$

Hence, f is continuous at $x = 0$,

28. **Examine the function $f(x)$ defined as follows for continuity at $x = 0, 1, 2$. Also discuss the kind of discontinuity, if any.**

$$f(x) = \begin{cases} -x^2 & \textbf{if } x \leqslant 0 \\ 5x-4 & \textbf{if } 0 < x \leqslant 1 \\ 4x^2-3x & \textbf{if } 1 < x < 2 \\ 3x+4 & \textbf{if } x \geqslant 2 \end{cases}$$

At $\mathbf{x = 0}$

Here, $f(0) = 0$.
Now,

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (-x^2) = 0$$

and

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} 5x - 4 = -4$$

Since,

$$\lim_{x \to 0-} f(x) \neq \lim_{x \to 0+} f(x)$$

f is discontinuous at $x = 0$

Also, it is a discontinuity of first kind as the left-hand and right-hand limits both exist but they are not equal.

At x = 1
Here, $f(1) = 5(1) - 4 = 1$.
Now,

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} 5x - 4 = 5 - 4 = 1$$

and

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} 4x^2 - 3x = 4 - 3 = 1$$

Since,

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(1)$$

f is continuous at $x = 1$

At x = 2
Here, $f(2) = 3x + 4 = 3(2) + 4 = 10$. Also,

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} 4x^2 - 3x = 4(4) - 3(2) = 10$$

and

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} 3x + 4 = 3(2) + 4 = 10$$

Since,

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(2)$$

f is continuous at $x = 2$

29. **If $[x]$ denotes the largest integer less than or equal to x, then discuss the continuity at $x = 3$ for the function $f(x) = x - [x],\ \forall x \geqslant 0$,**

We have,

$$\lim_{x \to 3-} x - [x] = 3 - 2 = 1$$

and

$$\lim_{x \to 3+} x - [x] = 3 - 3 = 0$$

As,

$$\lim_{x \to 3-} x - [x] \neq \lim_{x \to 3+} x - [x]$$

$f(x)$ is not continuous at $x = 3$.

30. Prove that the function f defined on $\mathbb{R}$ as follows is discontinuous at every point.

$$f(x) = \begin{cases} 1 & \textbf{when x is irrational} \\ -1 & \textbf{when x is rational} \end{cases}$$

Proof

Here, f is defined by

$$f(x) = \begin{cases} 1 & \text{when x is irrational} \\ -1 & \text{when x is rational} \end{cases}$$

First, let a be a rational number. By the definition of f we have

$$f(a) = -1$$

We know that, in every interval there are infinite number of rationals as well as irrationals. Therefore, for each positive number n we can choose an irrational number a_n in $\left(a - \frac{1}{n}, a + \frac{1}{n}\right)$ so that,

$$|a_n - a| < \frac{1}{n}$$

As a_n is an irrational we have $f(a_n) = 1$.

Since, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, we must have $\lim\limits_{n\to\infty} a_n = a$

Now,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} 1 = 1$$

Therefore,

$$\lim_{n\to\infty} f(a_n) \neq f(a) \quad (\because f(a) = -1)$$

Hence, $\lim\limits_{n\to\infty} a_n = a$ but $\lim\limits_{n\to\infty} f(a_n) \neq f(a)$.

Therefore f is not continuous at any rational number.

Next, let a be an irrational number. By the definition of f we have

$$f(a) = 1$$

We know that, in every interval there are infinite number of rationals as well as irrationals. Therefore, for each positive number n we can choose a rational number a_n in $\left(a - \frac{1}{n}, a + \frac{1}{n}\right)$ so that,

$$|a_n - a| < \frac{1}{n}$$

As a_n is a rational we have $f(a_n) = -1$.

Again, as $\lim_{n\to\infty} \frac{1}{n} = 0$, we must have $\lim_{n\to\infty} a_n = a$
Now,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} (-1) = -1$$

Therefore,

$$\lim_{n\to\infty} f(a_n) \neq f(a) \quad (\because f(a) = 1)$$

Hence, $\lim_{n\to\infty} a_n = a$ but $\lim_{n\to\infty} f(a_n) \neq f(a)$.

Therefore f is not continuous at any irrational number also. Thus, f is not continuous at any real number.

31. Prove that the function f defined on $\mathbb{R}$ as follows is continuous only at $x = 0$.

$$f(x) = \begin{cases} x & \textbf{when x is irrational} \\ -x & \textbf{when x is rational} \end{cases}$$

Proof
Here, f is defined by

$$f(x) = \begin{cases} x & \text{when x is irrational} \\ -x & \text{when x is rational} \end{cases}$$

First, let a be a non-zero rational number. By the definition of f we have

$$f(a) = -a$$

We know that, in every interval there are infinite number of rationals as well as irrationals. Therefore, for each positive number n we can choose an irrational number a_n in $\left(a - \frac{1}{n}, a + \frac{1}{n}\right)$ so that,

$$|a_n - a| < \frac{1}{n}$$

As a_n is an irrational we have $f(a_n) = a_n$.

Since, $\lim_{n\to\infty} \frac{1}{n} = 0$, we must have $\lim_{n\to\infty} a_n = a$

Now,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_n = a$$

Therefore,

$$\lim_{n\to\infty} f(a_n) \neq f(a) \quad (\because f(a) = -a, a \neq 0)$$

Hence, $\lim_{n\to\infty} a_n = a$ but $\lim_{n\to\infty} f(a_n) \neq f(a)$.

Therefore f is not continuous at any non-zero rational number.

Next, let a be an irrational number. By the definition of f we have

$$f(a) = a$$

We know that, in every interval there are infinite number of rationals as well as irrationals. Therefore, for each positive number n we can choose a rational number a_n in $\left(a-\frac{1}{n}, a+\frac{1}{n}\right)$ so that,

$$|a_n - a| < \frac{1}{n}$$

As a_n is a rational we have $f(a_n) = -a_n$.

Again, as $\lim_{n\to\infty} \frac{1}{n} = 0$, we must have $\lim_{n\to\infty} a_n = a$
Now,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty}(-a_n) = -a$$

Therefore,

$$\lim_{n\to\infty} f(a_n) \neq f(a) \quad (\because f(a) = a)$$

Hence, $\lim_{n\to\infty} a_n = a$ but $\lim_{n\to\infty} f(a_n) \neq f(a)$.

Therefore f is not continuous at any irrational number also.

Finally, let us examine continuiy of f at $x = 0$
We have

$$f(0) = 0$$

Now, for every real x, we have

$$|f(x)| = |x|$$

Therefore, for any given $\epsilon > 0$ we can take $\delta = \epsilon$ so that

$$|f(x) - f(0)| < \epsilon \quad \text{whenever} \quad |x - 0| < \delta$$

Therefore,

$$\lim_{x\to 0} f(x) = f(0)$$

Hence, f is continuous at $x = 0$.

Thus, f is continuous at zero and at no other real numbers.

32. **Obtain the points of discontinuity of the function f, defined on $[0,1]$ as follows**
$f(0) = 0,\ f(\frac{1}{2}) = \frac{1}{2},\ f(1) = 1$ **and**

$$f(x) = \begin{cases} \frac{1}{2} - x & \text{if } 0 < x < \frac{1}{2} \\ \frac{3}{2} - x & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

Also examine the kind of discontinuities.

The function has changes in its polynomial rules at $0, \frac{1}{2}$ and 1. So, we shall examine the continuity at these points only.
At $\mathbf{x = 0}$
As 0 is the left boundary of the closed interval $[0,1]$, we need to examine only the right

continuity at 0. Here, $f(0) = 0$.
Now,

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+}\left(\frac{1}{2} - x\right) = \frac{1}{2}$$

Since,

$$\lim_{x\to 0+} f(x) \neq f(0)$$

f is not right continuous at $x = 0$

Also, it is a discontinuity of first kind from right as the right-hand limit exists but it is not equal to $f(0)$.

At $\mathbf{x = \frac{1}{2}}$

As $\frac{1}{2}$ is an interior point of the closed interval $[0, 1]$, we shall examine the left-hand as well as right-hand limits. Here, $f(\frac{1}{2}) = \frac{1}{2}$.
Now,

$$\lim_{x\to \frac{1}{2}-} f(x) = \lim_{x\to \frac{1}{2}-}\left(\frac{1}{2} - x\right) = \frac{1}{2} - \frac{1}{2} = 0$$

Also,

$$\lim_{x\to \frac{1}{2}+} f(x) = \lim_{x\to \frac{1}{2}+}\left(\frac{3}{2} - x\right) = \frac{3}{2} - \frac{1}{2} = 1$$

Since,

$$\lim_{x\to \frac{1}{2}-} f(x) \neq \lim_{x\to \frac{1}{2}+} f(x)$$

f is discontinuous at $x = \frac{1}{2}$ and the discontinuity is of first kind.

At $\mathbf{x = 1}$

As 0 is the right boundary of the closed interval $[0, 1]$, we need to examine only the left continuity at 1. Here, $f(1) = 1$.
Now,

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-}\left(\frac{3}{2} - x\right) = \frac{3}{2} - 1 = \frac{1}{2}$$

Since,

$$\lim_{x\to 1-} f(x) \neq f(1)$$

f is not left continuous at $x = 1$

Also, it is a discontinuity of first kind from left as the left-hand limit exists but it is not equal to $f(1)$.

33. Bounded Above Set.

A set S is said to be a bounded above set if there exists some real number k such that

$$x \leqslant k, \forall x \in S$$

Here, k is called an upper bound of S.

34. Bounded Below Set.

A set S is said to be a bounded below set if there exists some real number k such that

$$k \leqslant x, \forall x \in S$$

Here, k is called a lower bound of S.

35. Bounded Set.

A set S is said to be a bounded set if it is bounded below as well as bounded above set.

36. Bounded Function

A real valued function f is said to be bounded if its range is a bounded set.

37. Interior Point of a Set and Neighbourhood of a point

Interior Point of a Set and Neighbourhood of a point
A real number a is said to be an interior point of a set S if there exists some open interval I such that

$$a \in I \subset S$$

Here, S is called a **Neighbourhood** of a.

38. Limit Point of a Set

A real number ξ is said to be a limit point of a set S if every neighbourhood of ξ contains infinitely many points of S.

Remark:
Equivalently, we can say that ξ is a limit point of S if every neighbourhood of ξ contains **atleast** one point of S **other than** ξ.

39. Limit Point of a Sequence

A real number ξ is said to be a limit point of a sequence $\{s_n\}$ if every neighbourhood of ξ contains infinitely many TERMS of sequence $\{s_n\}$.

40. Show that a continuous function on a closed interval is bounded.

Proof
Let a function f be continuous on a closed interval $I = [a, b]$.

If possible suppose f is not bounded above.
Therefore, for any positive number G there must be some value of $f(x)$ exceeding that positive number. In other words for each $G > 0$ there must be some $x \in [a, b]$ such that

$$G < f(x)$$

Hence, for each positive integer n we can always find some $x_n \in [a, b]$ such that,

$$n < f(x_n)$$

Since, $\{x_n\}$ is a sequence of points in a closed interval $[a, b]$ it is bounded. Therefore, by the Bolzano-Weierstrass therorem for sequence, there is a limit point of the sequence , say, ξ. As the closed interval $[a, b]$ is a closed set, we must have

$$\xi \in [a, b]$$

Because ξ is a limit point of $\{x_n\}$, there must be a subsequence, say $\{x_{n_k}\}_{k=1}^{\infty}$, such that

$$\lim_{k\to\infty} x_{n_k} = \xi$$

As for each x_{n_k} we have $n_k < f(x_{n_k})$, it follows that

$$\lim_{k\to\infty} f(x_{n_k}) = \infty$$

Thus,

$$\lim_{k\to\infty} x_{n_k} = \xi \quad \text{but} \quad \lim_{k\to\infty} f(x_{n_k}) \neq f(\xi)$$

Hence, f is not continuous at $\xi \in [a, b]$.

This contradicts our assumption that f is continuous on $[a, b]$. Therefore our supposition that f is not bounded above is wrong. Therefore, f must be bounded above.

With similar arguments it can be shown that f must be bounded below also. Hence, if a function is continuous on a closed interval then it is bounded.

41. **If a function is continuous on a closed interval $[a, b]$, then it attains its bounds at least once in $[a, b]$.**

Let f be a continuous function on a closed interval $[a, b]$.
If f is a constant function then clearly its bounds are equal to the constant value assumed by the function. Hence the bounds are attained at every point of $[a, b]$.

Now, suppose f is not a constant function. As f is continuous on $[a, b]$ it is bounded. Let m and M are the infimum and the supremum of f.

If possible suppose f does not attain its supremum at any point of $[a, b]$. Therefore for every

$x \in [a,b]$ we have $f(x) < M$.
Therefore,

$$0 < M - f(x)$$

Define,

$$g(x) = \frac{1}{M - f(x)}, \forall x \in [a,b]$$

As f is continuous on $[a,b]$, function $g(x)$ is also continuous on $[a,b]$, hence bounded also. Suppose, k is the supremum of $g(x)$. Therefore,

$$g(x) < k, \forall x \in [a,b]$$

Now,

$$\begin{aligned} g(x) < k &\Rightarrow \frac{1}{M - f(x)} < k \\ &\Rightarrow \frac{1}{k} < M - f(x) \\ &\Rightarrow f(x) < M - \frac{1}{k} \end{aligned}$$

But, $f(x) < M - \frac{1}{k}, \forall x \in [a,b]$ implies that $M - \frac{1}{k}$ is an upper bound of f, which is less than its supremum. This is not possible as no upper bound can be less than the supremum.

Therefore, our supposition that f does not attain its supremum at any point of $[a,b]$ is wrong. Hence, there must be some $\alpha \in [a,b]$ at which,

$$f(\alpha) = M$$

Thus, there is atleast one point in $[a,b]$ at which f attains its supremum.

Similarly, it can be shown that f attains its infumum at atleast one point.

42. **If a function f is continuous at an interior point c of $[a,b]$ and $f(c) \neq 0$, then prove that, there exists $\delta > 0$ such that $f(x)$ has the same sign as $f(c)$ for every $x \in (c-\delta, c+\delta)$.**

Let f be a function such that $f(c) \neq 0$ at an interior point c of $[a,b]$.
If f is continuous at c then for any given $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \quad \text{whenever} \quad |x - c| < \delta$$

Equivalently,

$$f(c) - \epsilon < f(x) < f(c) + \epsilon \quad \text{whenever} \quad x \in (c-\delta, c+\delta) \text{ - - - } (1)$$

As $f(c) \neq 0$, either $f(c) > 0$ or $f(c) < 0$.
If $f(c) > 0$ then taking ϵ such that $0 < \epsilon < f(c)$ we get

$$0 < f(c) - \epsilon$$

From (1) it follows that when $0 < f(c)$,

$$0 < f(x) \quad \text{whenever} \quad x \in (c-\delta, c+\delta)$$

Also, if $f(c) < 0$ then $0 < -f(c)$.
Taking ϵ such that $0 < \epsilon < -f(c)$ we get,

$$f(c) + \epsilon < 0$$

From (1) it follows that when $f(c) < 0$,

$$f(x) < 0 \quad \text{whenever} \quad x \in (c-\delta, c+\delta)$$

Thus, in any case there exists some $\delta > 0$ such that $f(x)$ keeps the same sign as $f(c)$ for every $x \in (c-\delta, c+\delta)$

43. If a function f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ are of opposite signs, then prove that there exists at least one point $\alpha \in (a,b)$ such that $f(\alpha) = 0$.

Here f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ are of opposite signs.
Without loss of generality, let us assume $f(a) > 0$ and $f(b) < 0$
Define a set S by

$$S = \{x/x \in [a,b] \text{ and } f(x) > 0\}$$

Clearly S is bounded above by b. Also as $f(a) > 0$ we have $a \in S$

Since, S is a non-empty and bounded subset of R, by order-completeness of R, the set S has the supremum in R, say α. Clearly $\alpha \in [a,b]$

Now, we shall prove that $\alpha \in (a,b)$ and $f(\alpha) = 0$

First we prove that $\alpha \in (a,b)$ by showing that $\alpha \neq a$ and $\alpha \neq b$.

Since, $f(a) > 0$ and f is right-continuous at a, there exists some δ_1 such that

$$f(x) > 0, \qquad \forall x \in (a, a+\delta_1)$$

Therefore, as $\alpha = sup.S$ we must have $a + \delta_1 < \alpha$.
Hence,

$$\alpha \neq a$$

Also, $f(b) < 0$ and f is left-continuous at b, there exists some δ_2 such that

$$f(x) < 0, \qquad \forall\, x \in (b-\delta_2, b)$$

Therefore, as $\alpha = sup.S$ we must have $\alpha < b - \delta_2$.
Hence,

$$\alpha \neq b$$

As $\alpha \neq a$ and $\alpha \neq b$ we must have

$$\alpha \in (a,b)$$

Finally, we show that $f(\alpha) = 0$.

If $f(\alpha) > 0$ then as f is continuous at the interior point α of (a, b) there is some $\delta > 0$ such that

$$f(x) > 0, \qquad \forall\, x \in (\alpha - \delta, \alpha + \delta)$$

If we choose, some δ_3 such that $0 < \delta_3 < \delta$ then

$$\alpha + \delta_3 \in \big(\alpha - \delta, \alpha + \delta\big)$$

Therefore,

$$f\big(\alpha + \delta_3\big) > 0$$

As $f\big(\alpha + \delta_3\big) > 0$ and $\alpha + \delta_3 \in [a, b]$ we have

$$\alpha + \delta_3 \in S$$

This is not possible as $\alpha < \alpha + \delta_3$ and α is the supremum of S so no member greater than α can be a member of S.

Therefore, our supposition that $f(\alpha) > 0$ is wrong. Hence we have

$$\mathbf{f}(\alpha) \not> \mathbf{0}$$

If $f(\alpha) < 0$ then as f is continuous at the interior point α of (a, b) there is some $\delta > 0$ such that

$$f(x) < 0, \qquad \forall\, x \in (\alpha - \delta, \alpha + \delta)$$

As α is the supremum of S there exists some $\beta \in S$ such that

$$\alpha - \delta < \beta < \alpha$$

Since, $\beta \in (\alpha - \delta, \alpha + \delta)$ we must have $f(\beta) < 0$

This is not possible as $\beta \in S$ implies that $f(\beta) > 0$.

Therefore, our supposition that $f(\alpha) < 0$ is also wrong. Hence,

$$\mathbf{f}(\alpha) \not< \mathbf{0}$$

As, we have $\mathbf{f}(\alpha) \not> \mathbf{0}$ and $\mathbf{f}(\alpha) \not< \mathbf{0}$ by the Law of Trichotomy of the ordered field R, we get,

$$\boxed{\mathbf{f}(\alpha) = \mathbf{0}}$$

44. **If a function f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then prove that it assumes every value between $f(a)$ and $f(b)$.**

If f is a constant function on $[a, b]$ then clearly it always assumes a fixed real number which is the infimum and supremum both. Therefore, the bounds of f are assumed by all the members of $[a, b]$.

Next, suppose f is continuous on $[a, b]$ and $f(a) \neq f(b)$. Consider any A between $f(a)$ and $f(b)$.

Clearly $f(a) - A$ and $f(b) - A$ have opposite signs.

Define a function $\phi(x)$ on $[a, b]$ by,

$$\phi(x) = f(x) - A, \ \forall x \in [a, b]$$

Here, $\phi(x)$ is continuous on $[a, b]$ as $f(x)$ is continuous on $[a, b]$.

Also, as $\phi(a) = f(a) - A$ and $\phi(b) = f(b) - A$, we have $\phi(a)$ and $\phi(b)$ of opposite signs.

Therefore, there must me some $c \in (a, b)$ such that

$$\phi(c) = 0$$

Therefore,

$$f(c) - A = 0$$

Hence,

$$f(c) = A \quad \text{for some } c \in [a, b]$$

45. If a function f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then prove that it assumes every value between its bounds.

Let f be continuous on $[a, b]$ and $f(a) \neq f(b)$. As f be continuous on $[a, b]$, it is bounded. Suppose m is the infimum and M is the supremum of f.

Also, f being continuous on $[a, b]$ it attains its bounds at some points in $[a, b]$. Let $\alpha, \beta \in [a, b]$ such that

$$f(\alpha) = M \quad \text{and} \quad f(\beta) = m$$

Since, f is continuous on $[a, b]$ it is also continuous on $[\alpha, \beta]$ or $[\beta, \alpha]$, depending on $\alpha < \beta$ or $\beta < \alpha$. Again, as f is continuous on $[\alpha, \beta]$ or $[\beta, \alpha]$ as the case may be, it assumes every value between $f(\alpha)$ and $f(\beta)$.

Thus, f assumes every value between its bounds M and m.

46. Uniform Continuity

Uniform Continuity:
A function f defined on an interval I is said to be uniformly continuous on I if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon, \ \forall x, y \in I \quad \text{for which} \quad |x - y| < \delta,$$

Remark:
The definition implies that for a given $\epsilon > 0$ existence of $\delta > 0$ may depend on ϵ but must be independent of choices of x and y

47. **Prove that if a function is uniformly continuous on an interval then it is continuous on that interval.**

Proof:
Let $f(x)$ be a uniformly continuous function on an interval I.
Therefore, for any given $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon, \ \forall x, y \in I \quad \text{for which} \quad |x - y| < \delta,$$

Consider any $c \in I$. Now for (1), we can arbitrariily choose x and y in I. So if we fix $y = c$, then for $\epsilon > 0$ there is some $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon, \forall x \in I \quad \text{such that} \quad |x - c| < \delta,$$

Therefore, f is continuous at any c in I.
Hence, f is continuous on I whenever it is uniformly continuous on I.

48. **Prove that if a function is continuous on a closed interval then it is also uniformly continuous on that interval.**

Proof:
Let $f(x)$ be a continuous function on a closed interval $[a, b]$.
If possible, suppose f is not uniformly continuous on $[a, b]$. Therefore, there is some $\epsilon > 0$ such that for every $\delta > 0$ there exist $x, y \in [a, b]$ so that

$$|f(x) - f(y)| \geqslant \epsilon, \quad \text{when} \quad |x - y| < \delta,$$

In particular, for each postive integer n taking $\delta = \frac{1}{n}$, there exist some x_n and y_n in $[a, b]$ such that,

$$|f(x_n) - f(y_n)| \geqslant \epsilon, \quad \text{when} \quad |x_n - y_n| < \frac{1}{n} \text{ - - - } (1)$$

Now, as the sequences $\{x_n\}$ and $\{y_n\}$ of points in $[a, b]$ are bounded by the Bolzano-Weierstrass theorem for sequences both the sequences have limit points.

Suppose ξ is a limit point of $\{x_n\}$ and η is a limit point of $\{y_n\}$. So, corresponding to ξ there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k \to \infty} x_{n_k} = \xi$$

Similarly, corresponding to η there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\lim_{k \to \infty} y_{n_k} = \eta$$

Also, from (1) it follows that,

$$|f(x_{n_k}) - f(y_{n_k})| \geqslant \epsilon, \quad \text{when} \quad |x_{n_k} - y_{n_k}| < \frac{1}{n_k}$$

As, $\frac{1}{n_k} \leqslant \frac{1}{k}$, and

$$\lim_{k\to\infty} \frac{1}{k} = 0$$

it follows that

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k}$$

Hence,

$$\xi = \eta$$

As $|f(x_n) - f(y_n)| \geqslant \epsilon$ it follows that even if $\lim\limits_{n\to\infty} f(x_{n_k})$ and $\lim\limits_{n\to\infty} f(y_{n_k})$ both exist, we have

$$\lim_{k\to\infty} f(x_{n_k}) \neq \lim_{k\to\infty} f(y_{n_k})$$

Thus, we have two sequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ converging to same limit ξ but $\lim\limits_{k\to\infty} f(x_{n_k})$ and $\lim\limits_{k\to\infty} f(y_{n_k})$ do not converge to same limits, if they exist.

But then f is not continuous at ξ. This is not possible as f is contunuous on $[a, b]$.

Due to this contradiction, we conclude that our supposition, that f is not uniformly continuous on $[a, b]$, is wrong.

Hence, f is uniformly continuous on $[a, b]$ whenever it is continuous on $[a, b]$

49. **Prove that $f(x) = \dfrac{1}{x}$ is not uniformly continuous on $(0, 1]$.**

Proof:

Clearly, $f(x)$ is continuous on $(0, 1]$.

If possible, suppose $f(x) = \dfrac{1}{x}$ is uniformly continuous on $(0, 1]$.

Therefore, for any given $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon, \ \forall x, c \in (0, 1] \quad \text{for which} \quad |x - c| < \delta,$$

Therefore,

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \epsilon, \ \forall x, c \in (0, 1] \quad \text{for which} \quad x \in (c - \delta, c + \delta)$$

Therefore,

$$\left|\frac{c - x}{cx}\right| < \epsilon, \ \forall x, c \in (0, 1] \quad \text{for which} \quad x \in (c - \delta, c + \delta)$$

Hence, by taking $c = \delta$ we get,

$$\left|\frac{\delta - x}{\delta x}\right| < \epsilon, \ \forall x \in (0, 1] \quad \text{for which} \quad x \in (0, 2\delta)$$

Now,

$$\frac{\delta - x}{\delta x} \to \infty \text{ as } x \to 0+$$

By taking x sufficiently close to 0 we can make $\frac{\delta-x}{\delta x}$ as large as we want. But, in that case condition (1) cannot be satisfied.

Therefore, our supposition that f is uniformly continuous on $(0, 1]$ is wrong.

Hence, $f(x) = \frac{1}{x}$ cannot be unformly continuous on $(0, 1]$.

50. Show that $f(x) = x^2$ is uniformly continuous on [-1,1].

Proof:
For any $x_1, x_2 \in [-1, 1]$ we have,

$$\begin{aligned} |f(x_1) - f(x_2)| &= \left|x_1^2 - x_2^2\right| \\ &= |(x_1 + x_2)(x_1 - x_2)| \\ &\leqslant |(1+1)(x_1 - x_2)| \qquad (\because x_1 \leqslant 1, x_2 \leqslant 1) \\ &= 2\,|x_1 - x_2| \end{aligned}$$

Thus, for any $x_1, x_2 \in [-1, 1]$, we have,

$$|f(x_1) - f(x_2)| \leqslant 2\,|x_1 - x_2| \qquad \text{- - - (1)}$$

Therefore, for any given $\epsilon > 0$ we can take $\delta = \dfrac{\epsilon}{2}$, so that,

$$\begin{aligned} |x_1 - x_2| < \delta \Rightarrow |x_1 - x_2| &< \frac{\epsilon}{2} \\ \Rightarrow 2\,|x_1 - x_2| &< \epsilon \\ \Rightarrow |f(x_1) - f(x_2)| &< \epsilon \quad \text{From (1)} \end{aligned}$$

Therefore, we can say that for every $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \epsilon\ \forall x_1, x_2 \in [-1, 1], \text{ for which } |x_1 - x_2| < \delta$$

Hence, f is continuous on $[-1, 1]$

51. Show that $f(x) = x^2$ is uniformly continuous on [1,2].

Proof:
For any $x_1, x_2 \in [1, 2]$ we have,

$$\begin{aligned} |f(x_1) - f(x_2)| &= \left|x_1^2 - x_2^2\right| \\ &= |(x_1 + x_2)(x_1 - x_2)| \\ &\leqslant |(2+2)(x_1 - x_2)| \qquad (\because x_1 \leqslant 2, x_2 \leqslant 2) \\ &= 4\,|x_1 - x_2| \end{aligned}$$

Thus, for any $x_1, x_2 \in [-1, 1]$, we have,

$$|f(x_1) - f(x_2)| \leqslant 4|x_1 - x_2| \quad \text{- - - (1)}$$

Therefore, for any given $\epsilon > 0$ we can take $\delta = \frac{\epsilon}{4}$, so that,

$$\begin{aligned} |x_1 - x_2| < \delta &\Rightarrow |x_1 - x_2| < \frac{\epsilon}{4} \\ &\Rightarrow 4|x_1 - x_2| < \epsilon \\ &\Rightarrow |f(x_1) - f(x_2)| < \epsilon \quad \text{From (1)} \end{aligned}$$

Therefore, we can say that for every $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \epsilon \; \forall x_1, x_2 \in 1, 2], \text{ for which } |x_1 - x_2| < \delta$$

Hence, f is continuous on $[1, 2]$

52. Derivative of a function at a point

Derivative of a function:
A real valued function f, defined on an interval $I = [a, b]$, is said to be derivable or differentiable at an interor point c of I if the following limit exists

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

The limit is called the Derivative or Differential Coefficient of f at c and it is generally denoted by $\mathbf{f'(c)}$. Also the process of finding the derivaive is called DIFFERENTIATION.

Remark:
Above limit in the definition can be equvalently evaluated using

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

53. Left Hand Derivative

Left Hand Derivative:
A real valued function f, defined on an interval $I = [a, b]$, is said to be derivable or differentiable from left at a point c if the following limit exists

$$\lim_{x \to c-} \frac{f(x) - f(c)}{x - c}$$

The limit is called the Left Hand Derivative of f at c and it is generally denoted by $\mathbf{f'(c-)}$ or $Lf'(c)$.

54. Right Hand Derivative

Right Hand Derivative:

A real valued function f, defined on an interval $I = [a, b]$, is said to be derivable or differentiable from right at a point c if the following limit exists

$$\lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$$

The limit is called the Right Hand Derivative of f at c and it is generally denoted by $\mathbf{f'(c+)}$ or $Rf'(c)$.

55. Derivability of a function on an open interval

Derivability of a function on an open interval:

A real valued function f, defined on an open interval (a, b), is said to be derivable on the interval if it is derivable at every point $c \in (a, b)$.

In other words if the following limit $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exits at every point $c \in (a, b)$ then f is called derivale on (a, b).

56. Derivability of a function on a closed interval

Derivability of a function on a closed interval:

A real valued function f, defined on a closed interval $[a, b]$, is said to be derivable on the interval if

(i) it is derivable at every point $c \in (a, b)$.

(ii) if it is Right derivable at a

(iii) if it is Left derivable at b

57. Show that $f(x) = x^2$ is derivable on [0,1].

Given function is $f(x) = x^2$.

Derivability at any c $\in (0, 1)$

$$\begin{aligned} \lim_{x \to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \to c} \frac{x^2 - c^2}{x - c} \\ &= \lim_{x \to c} \frac{(x + c)(x - c)}{x - c} \\ &= \lim_{x \to c} (x + c) \\ \therefore \lim_{x \to c} \frac{f(x) - f(c)}{x - c} &= 2c \end{aligned}$$

As $\lim_{x \to 1} \frac{f(x) - f(c)}{x - c}$ exists f is derivable at every $c \in (0., 1)$.

Right derivability at 0

$$\begin{aligned}\lim_{x\to0+}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to0+}\frac{x^2-0}{x-0}\\ &= \lim_{x\to0+} x\\ \therefore \lim_{x\to0+}\frac{f(x)-f(0)}{x-0} &= 0\end{aligned}$$

As $\lim\limits_{x\to0+}\dfrac{f(x)-f(0)}{x-0}$ exists f is Right derivable at 0.

Left derivability at 1

$$\begin{aligned}\lim_{x\to1-}\frac{f(x)-f(1)}{x-1} &= \lim_{x\to1-}\frac{x^2-1}{x-1}\\ &= \lim_{x\to1-}(x+1)\\ \therefore \lim_{x\to1-}\frac{f(x)-f(1)}{x-1} &= 2\end{aligned}$$

As $Rf'(0), Lf'(1)$ and $f'(c), \forall c\in(0,1)$ exist, $f(x)=x^2$ is derivable on $[0,1]$.

58. **At $x=1$ examine the derivability of $f(x)=\begin{cases} x & \text{if } 0\leqslant x<1\\ 1 & \text{if } x\geqslant 1\end{cases}$**

Given function is

$$f(x)=\begin{cases} x & \text{if } 0\leqslant x<1\\ 1 & \text{if } x\geqslant 1\end{cases}$$

Now,

$$\begin{aligned}\lim_{x\to1-}\frac{f(x)-f(1)}{x-1} &= \lim_{x\to1+}\frac{x-1}{x-1}\\ &= 1\\ \therefore Lf'(1) &= 1\end{aligned}$$

Also,

$$\begin{aligned}\lim_{x\to1+}\frac{f(x)-f(1)}{x-1} &= \lim_{x\to1+}\frac{1-1}{x-1}\\ &= \lim_{x\to1+} 0\\ &= 0\\ \therefore Rf'(1) &= 0\end{aligned}$$

Since, $Lf'(1)\neq Rf'(1)$ function f is not derivable at 1.

59. Show that the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 \quad \text{if } x = 0 \end{cases}$$

is derivable at $x = 0$ but $\lim_{x\to 0} f'(x) \neq f'(0)$

Given function is

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 \quad \text{if } x = 0 \end{cases}$$

Now,

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x} = \lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$$

As

$$\left| x \sin\left(\frac{1}{x}\right) - 0 \right| = |x| \left| \sin\left(\frac{1}{x}\right) \right| \leqslant |x|$$

for any given $\epsilon > 0$ we can take $\delta = \epsilon$ so that,

$$\left| x \sin\left(\frac{1}{x}\right) - 0 \right| < \epsilon \quad \text{whenever} \quad |x - 0| < \delta$$

Therefore, $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$,

Hence,

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 0$$

Now, for $x \neq 0$,

$$f'(x) = 2x \sin\frac{1}{x} + x^2 \left[\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)\right] = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$$

As, $x \to 0- \Rightarrow \frac{1}{x} \Rightarrow -\infty$ and $x \to 0+ \Rightarrow \frac{1}{x} \Rightarrow \infty$, none of $\sin\left(\frac{1}{x}\right)$ and $\cos\left(\frac{1}{x}\right)$ can tend to a fixed number.

Therefore, $\lim_{x\Rightarrow 0} f'(x)$ does not exist.

Hence,

$$\lim_{x\to 0} f'(x) \neq f'(0)$$

60. Prove that a function which is derivable at a point is necessarily continuous at that point. Is the converse true? Justify.

Let f be a derivable function at a point c. The derivative $f'(c)$ is given by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{- - - (1)}$$

Now,

$$\begin{aligned}\lim_{x \to c} f(x) - f(c) &= \lim_{x \to c}[f(x) - f(c)] \\ &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \times (x - c) \\ &= f'(c)(c - c) \\ &= f'(c)(0) \\ &= 0 \\ \therefore \lim_{x \to c} f(x) &= f(c)\end{aligned}$$

Hence f is continuous at c.

Next we show that if a function is continuos at a point c then it is not necessarily differentiable at c. Consider the function $f(x) = |x|$.

Here,

$$f(0) = 0$$

Now,

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} |x| = \lim_{x \to 0}(-x) = 0$$

and

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} |x| = \lim_{x \to 0} x = 0$$

Since,

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(0)$$

$f(x) = |x|$ is continuous at $x = 0$

Now, let us examine the derivability of $f(x)$ at $x = 0$.

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{|x| - 0}{x} = \lim_{x \to 0-} \frac{-x}{x} = \lim_{x \to 0-} -1 = -1$$

Therefore,

$$Lf'(0) = -1$$

Also,

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{|x| - 0}{x} = \lim_{x \to 0+} \frac{x}{x} = \lim_{x \to 0+} 1 = 1$$

Therefore,

$$Rf'(0) = 1$$

Since, $Lf'(0) \neq Rf'(0)$ function f is not derivable at $x = 0$.

Thus, the function f is continuous at 0 but it is not differentiable at 0.

Hence, the converse is not true.

61. **If f is a derivable at c and $f(c) \neq 0$ then the function $\frac{1}{f}$ is also derivable at c and**

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{\{f(c)\}^2}$$

Proof:
Let f be a derivable function at a point c. Now, the derivative $f'(c)$ is given by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{- - - (1)}$$

As f is derivable at c it is continuous also at c. Also if $f(c) \neq 0$ then for some neighbourhood N of c

$$f(x) \neq 0, \ \forall x \in N$$

Therefore for $x \in N$,

$$\frac{\frac{1}{f(x)} - \frac{1}{f(c)}}{x - c} = \frac{f(c) - f(x)}{(x - c)f(c)f(x)} = -\frac{f(x) - f(c)}{x - c} \frac{1}{f(x)f(c)}$$

$$\begin{aligned}\lim_{x \to c} \frac{\frac{1}{f(x)} - \frac{1}{f(c)}}{x - c} &= -\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \frac{1}{f(x)f(c)} \\ &= -f'(c) \times \frac{1}{f(c).f(c)} \\ &= -\frac{f'(c)}{\{f(c)\}^2}\end{aligned}$$

Hence, $\frac{1}{f}$ is derivable at c and

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{\{f(c)\}^2}$$