USO5CMTH21 UNIT-4 SERIES

INFINITE SERIES

Defn: Series: A series is the sum of the terms of a sequence.

Thus if $a_1, a_2, \ldots a_n, \ldots$ is a seq. then the sum $a_1 + a_2 + \ldots$ of all the terms is called an infinite series or series.

It is denoted by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$

Defn Partial sum of series:-

$S_n = \sum_{i=1}^{n} a_i$ is called the n^{th} partial sum

Defn Seq. of partial sum:

Let $S_n = \sum_{i=1}^{n} a_i$, then $\{S_n\}$ is called the seq. of partial sums

Remarks (i) Partial sums $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, ...

(ii) A series is said to be cgt., divergent or oscillate iff its seq. of partial sums $\{S_n\}$ cgt., dist. or oscillate resp.

Scanned with CamScanner

Thm.2 P.T. a necessary condition for convergence of an infinite series $\sum a_n$ is that $\lim a_n = 0$

Proof: Let $S_n = a_1 + a_2 + \cdots + a_n$ & $\{S_n\}$ be the seq. of partial sums

If series $\sum a_n$ is cgt. then seq. $\{S_n\}$ is also cgt.

Let $\lim_{n\to\infty} S_n = S$, say

W.K.T $S_n = a_1 + a_2 + \cdots + a_n$

$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$

$\therefore S_n - S_{n-1} = a_n \quad \forall n > 1$

$\therefore n \to \infty [S_n - S_{n-1}] = n \to \infty a_n$

$\Rightarrow n \to \infty S_n - n \to \infty S_{n-1} = n \to \infty a_n$

$\Rightarrow S - S = n \to \infty a_n$

$\Rightarrow 0 = n \to \infty a_n$

Hence $\lim_{n\to\infty} a_n = 0$

Remark: (i) From above thm we say that If $\lim a_n \neq 0$ then series $\sum a_n$ is not cgt.

(ii) Converse of above thm need not be true

$\because$ For $\sum \frac{1}{n}$, $a_n = \frac{1}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

but $\sum \frac{1}{n}$ is not cgt.

Scanned with CamScanner

Ex. S.T. the series $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$ is not cgt.

Soln: Here $u_n = \frac{n}{n+1}$

$\therefore \lim u_n = \lim \frac{n}{n+1} = \lim \frac{n}{n\left[1+\frac{1}{n}\right]} = 1 \neq 0$

$\therefore$ Series is not cgt.

Thm: 2 State & Prove Cauchy's General principle of convergence for series

Statement: A series $\sum u_n$ is cgt. iff for each $\varepsilon > 0$ $\exists$ +ve integer m $\ni$ $|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \varepsilon$

$\forall\ n \geq m,\ p \geq 1$

i.e $|S_{n+p} - S_n| < \varepsilon\ \forall\ n \geq m,\ p \geq 1$

Proof: W.K.T $\sum u_n$ is cgt

iff. seq. $\{S_n\}$ of partial sums is cgt.

iff for each $\varepsilon > 0$ $\exists$ +ve integer m $\ni$ $|S_{n+p} - S_n| < \varepsilon\ \forall\ n \geq m$ & $p \geq 1$

(∵ by cauchy general principle of cgt. for seq.)

iff for each $\varepsilon > 0$ $\exists$ +ve integer m $\ni$ $|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \varepsilon$,

$\forall\ n \geq m$ & $p \geq 1$

Hence thm. is proved.

Scanned with CamScanner

Ex. P.T $\sum \frac{1}{n}$ does not cgt.

Soln: For $\varepsilon = \frac{1}{4}$, $n = m$, $p = m$

we say that

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}|$$

$$= |u_{m+1} + u_{m+2} + \cdots + u_{2m}|$$

$$= \left|\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m}\right| \quad (\because \forall u_i > 0)$$

$$> \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m} \quad (m \text{ times})$$

$$= m \cdot \frac{1}{2m} \quad \left(\because \begin{matrix} m+1 < 2m \\ \cdots\cdots \end{matrix} \Rightarrow \frac{1}{m+1} > \frac{1}{2m}\right)$$

$$= \frac{1}{2} > \varepsilon \quad (\because \varepsilon = 1/4)$$

Thus $|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| > \varepsilon$

$\therefore$ by a cauchy General principle we say that $\sum \frac{1}{n}$ is not cgt.

Thm: 3 If $\sum u_n = u$ then P.T

i) $\sum_{n=1}^{n} c u_n = c \sum_{n=1}^{n} u_n$, where c is any no. independent of n.

ii) $\sum_{n=0}^{\infty} u_n = u + u_0$.

iii) $\sum_{n=2}^{\infty} u_n = u - u_1$

Scanned with CamScanner

Proof: (i) W.K.T

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

$$\Rightarrow \sum_{n=1}^{n} c a_n = c \sum_{n=1}^{n} a_n$$

$$\Rightarrow \lim_{n\to\infty} \sum_{n=1}^{n} c a_n = \lim_{n\to\infty} \left[c \sum_{n=1}^{\infty} a_n \right]$$

$$\Rightarrow \sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n = cA$$

ii) Let $S_n = \sum_{n=1}^{n} a_n$, $S_n' = \sum_{n=0}^{n} a_n$

then $S_n' = a_0 + S_n$

$$\Rightarrow \lim_{n\to\infty} S_n' = \lim_{n\to\infty} [a_0 + S_n]$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n = a_0 + A$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n = a_0 + A$$

iii) Let $S_n = \sum_{n=1}^{n} a_n$, $S_n'' = \sum_{n=2}^{n} a_n$

then $S_n'' = S_n - a_1$

$$\Rightarrow \lim_{n\to\infty} S_n'' = \lim_{n\to\infty} [S_n - a_1]$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - a_1 = A - a_1$$

Scanned with CamScanner

$\Rightarrow \sum_{n=2}^{\infty} u_n = u - u_1$

Remark:

i) If $\sum_{n=1}^{\infty} u_n$ is dgt then $\sum_{n=1}^{\infty} cu_n$, $\sum_{n=0}^{\infty} u_n$ or $\sum_{n=2}^{\infty} u_n$ is also dgt

ii) The behaviour of a series (cgt or dgt) is not altered by
   I) the alteration, addition or substraction of a finite no. of terms of
   II) Multi. of all the terms by a finite nonzero nos.

Thm. 4 If $\sum u_n = u$, $\sum v_n = v$ then P.T $\sum \omega_n = u \pm v$, where $\omega_n = u_n \pm v_n$, $\forall n$

Proof: W.K.T

$$\sum_{s=1}^{n} \omega_s = \sum_{s=1}^{n} (u_s \pm v_s)$$

$$\Rightarrow \sum_{s=1}^{n} \omega_s = \sum_{1}^{n} u_s \pm \sum_{1}^{n} v_s$$

$$\Rightarrow \lim_{n\to\infty} \sum_{1}^{n} \omega_s = \lim_{n\to\infty} \left[\sum_{1}^{n} u_s \pm \sum_{1}^{n} v_s \right]$$

$$\Rightarrow \sum_{1}^{\infty} \omega_s = \sum_{1}^{\infty} u_s \pm \sum_{1}^{\infty} v_s$$

$$\Rightarrow \sum_{n=1}^{\infty} \omega_n = u \pm v$$

Scanned with CamScanner

Thm:5 If $\sum u_n$ converges to u then p.t. any series obtained from $\sum u_n$ by grouping the terms in brackets without altering the order of the terms is also converges to u.

OR p.t a series obtained from a given cgt. series by a grouping of terms is also cgt. to the same limit.

Rmrk. Does the converse hold! Verify it.

Proof: Let $\sum u_n$ be series & let $\sum v_n$ be the series derived from $\sum u_n$ by insertion of brackets

Let $\{S_n\}$ & $\{G_n\}$ be the seq. of partial sums of $\sum u_n$ & $\sum v_n$ resp.

W.K.T $\sum u_n = u$

$\therefore$ seq. $\{S_n\}$ converges to u

$\therefore$ seq. $\{G_n\}$ converges to u

($\because$ subseq. of cgt. seq. is also cgt.)

$\therefore$ $\sum v_n = u$ i.e. $\sum v_n$ converges to u.

Scanned with CamScanner

2:

Converse: Converse need not be true.

For $\sum u_n = \sum (-1)^{n+1}$

$= 1-1+1-1+1-1$

Let

$\sum v_n = (1-1)+(1-1)+(1-1)+\cdots$

Then $\sum v_n$ converges to 0

but $\sum u_n$ is not cgt.

$\left(\because \sum u_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases} \right)$

Scanned with CamScanner

Thm: 6 P.T a +ve term series is either cgt. or dgt. to $+\infty$.

Proof: Let $\sum u_n$ be any +ve term series & $\{S_n\}$ be seq. of its partial sums

$\therefore S_n = u_1 + u_2 + \cdots + u_n \geq 0 \quad \forall n$

$S_{n-1} = u_1 + u_2 + \cdots + u_{n-1}$

$\Rightarrow S_n - S_{n-1} = u_n \geq 0 \quad \forall n$

$\Rightarrow S_n \geq S_{n-1} \quad \forall n$

$\Rightarrow \{S_n\}$ is mono. increasing seq.

W.K.T mono. increasing seq. is either cgt or dgt but can not oscillate

$\therefore \{S_n\}$ is either cgt or dgt to ∞

$\therefore \sum u_n$ is either cgt or dgt to $+\infty$.

Thm: 7 P.T. a +ve term series is cgt. iff the seq. of its partial sums is bounded above.

Proof: Let $\sum u_n$ be any +ve term series & $\{S_n\}$ be the seq. of its partial sums

$\therefore S_n = u_1 + u_2 + \cdots + u_n, \quad \forall n$

$S_{n-1} = u_1 + u_2 + \cdots + u_{n-1}$

Scanned with CamScanner

$\therefore S_n - S_{n-1} = u_n \geq 0$

$\Rightarrow S_n \geq S_{n-1} \quad \forall n$

$\Rightarrow$ Seq. $\{S_n\}$ is monotonic increasing

~~$\Rightarrow \{S_n\}$ is bdd above~~

Also W.K.T mono. increasing seq. is cgt. iff it is bdd above

$\therefore \{S_n\}$ is cgt iff it is bdd above

$\therefore \sum u_n$ is cgt. iff the seq. $\{S_n\}$ of its partial sums is bdd. above.

Remark: i) The seq. of partial sums of a series of -ve term is mono. decreasing seq.

ii) A series of -ve term is cgt. iff the seq. of its partial sums is bdd below

iii) A series of -ve terms is either cgt. or diverges to $-\infty$.

Scanned with CamScanner

Thm: P.T. the +ve term geometric series $1+x+x^2+\ldots$ (i.e $\sum_{n=0}^{\infty} x^n$) is cgt. for $x<1$ and dgt. to ∞ for $x \geq 1$

Proof: case(i) If $0 \leq x < 1$

Let $1+x+x^2+\ldots$ be given series & $\{S_n\}$ be its seq. of partial sums then $S_n = 1+x+x^2+\ldots+x^{n-1}$

$$\Rightarrow S_n = \frac{1-x^n}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x} \leq \frac{1}{1-x}$$

$$\Rightarrow S_n \leq \frac{1}{1-x} \quad \forall n$$

$\Rightarrow \{S_n\}$ is bdd above

$\therefore$ by above thm. we say that $1+x+x^2+\ldots$ is cgt.

case(ii) If $x=1$ then $S_n = n$

$\therefore \{S_n\}$ is not bdd. above

$\Rightarrow 1+x+x^2+\ldots$ is diverges to $+\infty$

case(iii) If $x>1$ then

$$S_n = 1+x+x^2+\ldots+x^{n-1}$$

$$> 1+1+1+\ldots+1 = n$$

$$\Rightarrow S_n > n \quad \forall n$$

$\therefore \{S_n\}$ is not bdd above

Scanned with CamScanner

∴ Given series $1+3+3^2+\dots$ is diverges to ∞.

Hence Geometric series is cgt. if $r<1$ & dgt. if $r \geq 1$.

Scanned with CamScanner

Thm:9 P.T. a +ve term series $\sum \frac{1}{n^p}$ is cgt iff $p>1$

Proof: Let S_n be the n^{th} partial sums of $\sum \frac{1}{n^p}$

i.e $S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}$

Case-(i) If $p>1$

Now

2^0 terms $\quad \frac{1}{1^p} = 1$

$(2^1$ terms$)$ $\quad \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}}$

$(2^2$ terms$)$ $\quad \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$

$$= \frac{4}{4^p} = \frac{1}{4^{p-1}} = \frac{1}{(2^{p-1})^2}$$

$\vdots$

$\left(\text{to reach } 2^n \text{ terms}\right)$

$$\frac{1}{(2^n)^p} + \frac{1}{(2^n+1)^p} + \frac{1}{(2^n+1)^p} + \cdots + \frac{1}{(2^{n+1}-1)^p}$$

$$< \frac{1}{(2^n)^p} + \frac{1}{(2^n)^p} + \cdots \frac{1}{(2^n)^p} = \frac{2^n}{(2^n)^p}$$

$$= \frac{1}{(2^n)^{p-1}} = \frac{1}{(2^{p-1})^n}$$

Scanned with CamScanner

By adding above results, we get

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots\cdots+\frac{1}{(2^{n+1}-1)^p}=S_{2^{n+1}-1}$$

$$\Rightarrow S_{2^{n+1}-1} < 1+\frac{1}{2^{p-1}}+\frac{1}{(2^{p-1})^2}+\cdots+\frac{1}{(2^{p-1})^n}$$

$$= \frac{1-\left(\frac{1}{2^{p-1}}\right)^{n+1}}{1-\frac{1}{2^{p-1}}} < \frac{1}{1-\frac{1}{2^{p-1}}} = \frac{2^{p-1}}{2^{p-1}-1}$$

Thus $S_{2^{n+1}-1} < \frac{2^{p-1}}{2^{p-1}-1} = \alpha$, say $\forall n$

Since given series $\sum \frac{1}{n^p}$ is +ve term series $\therefore \{S_n\}$ is an increasing seq.

$\therefore S_n > S_m \quad \forall n \geq m$

W.K.T $n < 2^n < 2^{n+1}-1$

$\therefore S_n < S_{2^n} < S_{2^{n+1}-1} \quad \forall n$

$\Rightarrow S_n < \alpha \quad \forall n \quad (\because S_{2^{n+1}-1} < \alpha)$

$\Rightarrow \{S_n\}$ is bdd above

$\therefore \{S_n\}$ is cgt.

$\therefore \sum \frac{1}{n^p}$ is cgt for $p>1$

Scanned with CamScanner

Case-(iii) If $p \leq 1$

W.K.T $n \in N$, $p \leq 1$ $\therefore n^p \leq n$

$$\Rightarrow \frac{1}{n^p} \geq \frac{1}{n} \quad \forall n \in N, p \leq 1$$

$$\therefore 1 + \frac{1}{2^p} \geq 1 + \frac{1}{2} > \frac{1}{2}$$

Total 2^{1-1} terms

$$\frac{1}{3^p} + \frac{1}{4^p} \geq \frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

Total 2^{2-1} terms

$$\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Total 2^{m-1} terms

$$\frac{1}{(2^{m-1}+1)^p} + \frac{1}{(2^{m-1}+2)^p} + \cdots + \frac{1}{(2^m)^p}$$

$$\geq \frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \cdots + \frac{1}{2^m}$$

$$> \frac{1}{2^m} + \frac{1}{2^m} + \cdots + \frac{1}{2^m} \quad (2^{m-1} \text{ times})$$

$$= \frac{2^{m-1}}{2^m} = \frac{1}{2}$$

By adding above results, we get

$$S_{2^m} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2^m)^p}$$

$$> \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \quad (m \text{ terms})$$

$$= m/2$$

Scanned with CamScanner

$\Rightarrow S_{2^m} > \frac{m}{2}$

Now we p.t $\{S_n\}$ is not bdd above

If G be any no. however large then $\exists\ m \in \mathbb{N}$ $\frac{m}{2} > G$

Let $n > 2^m$ then

$S_n > S_{2^m}$ ($\because \{S_n\}$ is increasing seq.)

$\Rightarrow S_n > \frac{m}{2} > G$

$\Rightarrow \{S_n\}$ is not bdd above

Thus $\{S_n\}$ of partial sums is not bdd above $\therefore$ It is not cgt

$\therefore \sum \frac{1}{n^p}$ is not cgt. for $p \leq 1$

Hence $\sum \frac{1}{n^p}$ is cgt iff $p > 1$.

Scanned with CamScanner

Thm: 10 State & Prove Comparision test of 1st type.

Statement: If $\sum u_n$ & $\sum v_n$ are two +ve term series, $k \neq 0$ a fixed +ve no. (independent of n) and $\exists$ a +ve integer $m \Rightarrow u_n \leq k v_n \ \forall\ n \geq m$ then P.T

(i) $\sum u_n$ is cgt., if $\sum v_n$ is cgt

(ii) $\sum v_n$ is dgt., if $\sum u_n$ is dgt.

Proof: Let $n \geq m$, s_n & t_n be the n^{th} partial sums of $\sum u_n$ & $\sum v_n$ resp.

then $s_n = u_1 + u_2 + \cdots + u_n$

$t_n = v_1 + v_2 + \cdots + v_n$

Now

$s_n - s_m = (u_1 + u_2 + \cdots + u_n) - (u_1 + u_2 + \cdots + u_m)$

$= u_{m+1} + u_{m+2} + \cdots + u_n$

$\leq k v_{m+1} + k v_{m+2} + \cdots + k v_n$

$= k [v_{m+1} + v_{m+2} + \cdots + v_n]$

$= k [t_n - t_m]$

$\Rightarrow s_n - s_m \leq k t_n - k t_m$

$\Rightarrow s_n \leq k t_n + [s_m - k t_m]$

$\Rightarrow s_n \leq k t_n + h$ ——— (1)

where $h = s_m - k t_m$ is finite

(i) If $\sum v_n$ is cgt, then seq. $\{t_n\}$ is cgt

$\therefore \{t_n\}$ is bdd above

Scanned with CamScanner

$\therefore \exists\ B \in R \ni t_n \leq B, \forall n$

$\therefore$ by (1), $s_n \leq kB + h, \forall n \geq m$

$\Rightarrow \{s_n\}$ is bdd above

$\Rightarrow \{s_n\}$ is cgt. [$\because$ Every mono. incre. bdd above seq. is cgt]

$\Rightarrow \sum u_n$ is cgt.

ii) If $\sum u_n$ is dgt, then seq. $\{s_n\}$ is not bdd. above.

$\therefore$ for any no. G, however large $\exists\ m_0 \in N \ni s_n > G\ \forall n \geq m_0$

Let $m_1 = \max\{m, m_0\}$

$\therefore$ by (1) $k t_n + h > G \quad \forall n \geq m_1$

$\Rightarrow t_n > \frac{1}{k}(G - h), k \neq 0, \forall n \geq m_1$

$\Rightarrow \{t_n\}$ is not bdd above seq.

$\therefore \{t_n\}$ is divergent

$\therefore \sum v_n$ is divergent.

Thm: 11 State & Prove Comparision test of 1st type in limit form.

Start: If $\sum u_n$ & $\sum v_n$ are two +ve term series $\ni \lim_{n\to\infty} \frac{u_n}{v_n} = l$, when l is non zero finite no. then P T the two series converge or diverge together

i.e. P T. the two series have same behaviours.

Scanned with CamScanner

Proof: W.K.T $a_n > 0,\ v_n > 0$

$\therefore \lim \frac{a_n}{v_n} = l > 0$



Let $\varepsilon > 0$ be such that $l - \varepsilon > 0$

Since $n \to \infty \ \frac{a_n}{v_n} = l$

$\therefore \exists\ m \in N \Rightarrow \left|\frac{a_n}{v_n} - l\right| < \varepsilon \quad \forall\ n \geq m$

$\therefore l - \varepsilon < \frac{a_n}{v_n} < l + \varepsilon$

$\Rightarrow (l-\varepsilon) v_n < a_n < (l+\varepsilon) v_n, \quad \forall n \geq m$ ——— (1)

If $\sum v_n$ is cgt then by (1)

$a_n < (l+\varepsilon) v_n \quad \forall n \geq m$

if $a_n < \alpha v_n$, $\alpha = l + \varepsilon$, $l \neq 0$

$\therefore$ by above thm. We say that $\sum a_n$ is cgt.

Also if $\sum v_n$ is dgt, then by (1)

$a_n > (l-\varepsilon) v_n$

if $a_n > k v_n \quad \forall n \geq m$, $k = l - \varepsilon > 0$

$\therefore$ By above thm we say that $\sum a_n$ is divergent

Similarly we can P.T. if $\sum a_n$ is cgt or dgt then $\sum v_n$ is also cgt or dgt resp.

Hence both series $\sum a_n$ & $\sum v_n$ have same behaviour.

Scanned with CamScanner

Thm: 12 State & Prove comparision test of second type.

Statement: If $\sum u_n$ & $\sum v_n$ are two +ve term series & $\exists$ a +ve integer $m \ni \frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}$, $\forall n \geq m$

then P.T.

(i) $\sum u_n$ is cgt, if $\sum v_n$ is cgt

(ii) $\sum v_n$ is dgt, if $\sum u_n$ is dgt.

Proof: For $n \geq m$, we have

$$\frac{u_m}{u_n} = \frac{u_m}{u_{m+1}} \cdot \frac{u_{m+1}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+3}} \cdots \cdots \frac{u_{n-1}}{u_n}$$

$$\geq \frac{v_m}{v_{m+1}} \cdot \frac{v_{m+1}}{v_{m+2}} \cdots \cdots \frac{v_{n-1}}{v_n} = \frac{v_m}{v_n}$$

$$\Rightarrow \frac{u_m}{u_n} \geq \frac{v_m}{v_n} \Rightarrow \frac{u_n}{u_m} \leq \frac{v_n}{v_m} \quad \forall n \geq m$$

$$\Rightarrow u_n \leq \left(\frac{u_m}{v_m}\right) v_n$$

$$\Rightarrow u_n \leq k v_n, \text{ where } k = \frac{u_m}{v_m} \text{ is fixed no.}$$

$\forall n \geq m$

$\therefore$ By comparision test of 1st type, we say that

$\sum u_n$ is cgt, if $\sum v_n$ is cgt &

$\sum v_n$ is dgt, if $\sum u_n$ is dgt.

Scanned with CamScanner

Thm:13 State & Prove Cauchy's root test.

Statement: If $\sum a_n$ is a +ve term series, such that $\lim_{n\to\infty}(a_n)^{1/n} = l$, then P.T.

(i) $\sum a_n$ is cgt. if $l < 1$

(ii) $\sum a_n$ is dgt., if $l > 1$

(iii) the test fails, if $l = 1$

Proof: Case (i): If $l < 1$

Let us select $\varepsilon > 0 \ni l + \varepsilon < 1$

Let $l + \varepsilon = \alpha$ then $\alpha < 1$

W.K.T $\lim_{n\to\infty}(a_n)^{1/n} = l$

$\therefore \exists\ m \in \mathbb{N} \ni |(a_n)^{1/n} - l| < \varepsilon, \ \forall\ n > m$

$\Rightarrow l - \varepsilon < (a_n)^{1/n} < l + \varepsilon$

$\Rightarrow (l-\varepsilon)^n < a_n < (l+\varepsilon)^n$

$\Rightarrow a_n < \alpha^n, \ \forall\ n \geq m, \ \alpha < 1$

W.K.T geometric series $\sum \alpha^n$ is cgt (∵ common ratio $\alpha < 1$)

$\therefore$ By comparision test, we say that $\sum a_n$ is cgt.

Case-(ii) If $l > 1$

Let us select $\varepsilon > 0 \ni l - \varepsilon > 1$

Let $l - \varepsilon = \beta$. then $\beta > 1$

Scanned with CamScanner

Since $\lim (a_n)^{1/n} = l$

$\therefore \exists\ m_1 \in \mathbb{N} \ni |(a_n)^{1/n} - l| < \varepsilon, \forall n > m_1$

$\Rightarrow l - \varepsilon < (a_n)^{1/n} < l + \varepsilon$

$\Rightarrow (l-\varepsilon)^n < a_n < (l+\varepsilon)^n, \forall n > m_1$

$\Rightarrow a_n > \beta^n \quad \forall n > m_1$

W.K.T Geometric series $\sum \beta^n$ is dgt

($\because$ common ratio $\beta > 1$)

$\therefore$ By Comparision test, we say that $\sum a_n$ is dgt.

Note: P.T. above test fails, if $l = 1$

Consider two series $\sum \frac{1}{n}$ & $\sum \frac{1}{n^2}$

such that $\lim \left(\frac{1}{n}\right)^{1/n} = 1$ & $\lim \left(\frac{1}{n^2}\right)^{1/n} = 1$

$\therefore l = 1$

but w.k.t $\sum \frac{1}{n}$ is dgt & $\sum \frac{1}{n^2}$ is cgt.

$\therefore$ above test fails, if $l = 1$.

Thm: 14 State & prove D'Alembert's ratio test

Statement:- If $\sum a_n$ is a +ve term series such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$ then P.T

(i) $\sum a_n$ is cgt, if $l < 1$

ii) $\sum a_n$ is dgt, if $l > 1$

iii) the test fails if $l = 1$.

Scanned with CamScanner

Proof: case-(i) If $l < 1$

Let us select $\varepsilon > 0 \ni l + \varepsilon < 1$

Let $l + \varepsilon = \alpha$. then $\alpha < 1$, $\alpha \neq 0$

W.K.T $\lim \frac{u_{n+1}}{u_n} = l$

$\therefore \exists\, m \in N \Rightarrow \left| \frac{u_{n+1}}{u_n} - l \right| < \varepsilon, \ \forall\, n \geq m$

$\Rightarrow l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon$ ——(1)

$\Rightarrow \frac{u_{n+1}}{u_n} < \alpha \quad , \forall\, n \geq m$

For $n \geq m$, W.K.T

$$\frac{u_n}{u_m} = \frac{u_{m+1}}{u_m} \cdot \frac{u_{m+2}}{u_{m+1}} \cdot \frac{u_{m+3}}{u_{m+2}} \cdots \frac{u_n}{u_{n-1}}$$

$$< \alpha \cdot \alpha \cdot \alpha \cdots \alpha \quad (n-m \text{ times})$$

$\Rightarrow \frac{u_n}{u_m} < \alpha^{n-m} \Rightarrow u_n < \left(\frac{u_m}{\alpha^m}\right) \alpha^n$

$\Rightarrow u_n < k\alpha^n, \ \forall\, n \geq m$, where $k = \frac{u_m}{\alpha^m}$ is fixed no.

Since Geometric Series $\sum \alpha^n$ is cgt ($\because$ common ratio $\alpha < 1$)

$\therefore$ By comparision test, we say that $\sum u_n$ is cgt.

Scanned with CamScanner

Case-ii) If $l > 1$

Let us select $\varepsilon > 0 \Rightarrow l - \varepsilon > 1$

Let $l - \varepsilon = \beta$ then $\beta > 1$

Since $\lim \frac{u_{n+1}}{u_n} = l$

$\therefore \exists\, m_1 \in \mathbb{N} \Rightarrow \left| \frac{u_{n+1}}{u_n} - l \right| < \varepsilon \quad \forall\, n > m_1$

$\Rightarrow l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon$

$\Rightarrow \frac{u_{n+1}}{u_n} > l - \varepsilon > 1 \quad \forall\, n > m_1$

for $n > m_1$,

$$\frac{u_n}{u_{m_1}} = \frac{u_{m_1+1}}{u_{m_1}} \cdot \frac{u_{m_1+2}}{u_{m_1+1}} \cdots \frac{u_n}{u_{n-1}}$$

$$> \beta \cdot \beta \cdots \beta \quad (n - m_1 \text{ times})$$

$\Rightarrow u_n > u_{m_1} \beta^{n - m_1}$

$\Rightarrow u_n > \left(\frac{u_{m_1}}{\beta^{m_1}} \right) \beta^n \quad \forall\, n > m_1$

$\Rightarrow u_n > K_1 \beta^n, \; n > m_1$, where

$K_1 = \frac{u_{m_1}}{\beta^{m_1}}$ is fixed no.

Since geometric series $\sum \beta^n$ is dgt

(∵ common ratio $\beta > 1$)

$\therefore$ By comparision test

$\sum u_n$ is dgt.

Scanned with CamScanner

NOTE: P.T D'Alembert test fails if $l=1$.

Consider $\sum \frac{1}{n}$ & $\sum \frac{1}{n^2}$ ∋

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{n}{n+1} = 1$$ &

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{n^2}{(n+1)^2} = 1 \quad \therefore l=1$$

But $\sum \frac{1}{n}$ is dgt & $\sum \frac{1}{n^2}$ is cgt.

∴ Test fails if $l=1$

Ex: S.T $1+\frac{1}{2!}+\frac{1}{3!}+\cdots$ is convergent.

Soln: W.K.T $3! > 2^2$, $4! > 2^3$, $5! > 2^4$, ...

$$\therefore 1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots < 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$$

$\Rightarrow u_n \le v_n \quad \forall\ n$

W.K.T geometric series $\sum v_n = 1+\frac{1}{2}+\frac{1}{2^2}+\cdots$ is cgt ($\because$ common ratio $r=\frac{1}{2}<1$)

∴ By Comparision test

$\sum u_n$ is cgt ∴ $1+\frac{1}{2!}+\frac{1}{3!}+\cdots$ is cgt.

Ex: S.T $\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \cdots$

diverges for $p>1$

Soln: W.K.T $\lim_{n\to\infty} \frac{(\log n)^p}{n} = 0$

$\therefore (\log n)^p < n \quad \forall\ n>1$

Scanned with CamScanner

$\therefore \frac{1}{(\log n)^p} > \frac{1}{n} \quad \forall n > 1$

Thus $a_n > v_n \quad \forall n > 1$

W.K.T $\sum v_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots = \sum \frac{1}{n+1}$

is dgt ($\because \sum \frac{1}{n^p}$ is dgt for $p = 1$)

$\therefore$ By Comparision test, we say that

$\sum a_n = \frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \cdots$ is dgt.

Ex. Prove $\frac{1 \cdot 2}{3^2 \cdot 4^2} + \frac{3 \cdot 4}{5^2 \cdot 6^2} + \frac{5 \cdot 6}{7^2 \cdot 8^2} + \cdots$ is cgt.

Soln:

Let $a_n = \frac{(2n-1)(2n)}{(2n+1)^2 (2n+2)^2}$ & $v_n = \frac{1}{n^2}$

Then $\lim \frac{a_n}{v_n} = \lim \frac{(2n-1)(2n)}{(2n+1)^2 (2n+2)^2} \cdot n^2$

$= \lim_{n \to \infty} \frac{2n^2 (2 - 1/n) \cdot n^2}{n^2 (2 + 1/n)^2 \, n^2 (2 + 2/n)^2}$

$= \frac{2(2-0)}{(2)^2 (2)^2} = \frac{4}{4.4} = \frac{1}{4} \neq 0$

$\therefore \sum a_n$ & $\sum v_n$ have same behaviours

W.K.T $\sum v_n = \sum \frac{1}{n^2}$ is cgt

($\because \sum \frac{1}{n^p}$ is cgt, if $p > 1$)

$\therefore \sum a_n$ is also cgt.

Scanned with CamScanner

Ex. Investigate the behaviours of the series whose n^{th} term is $\sin\frac{1}{n}$

Sol^n Let $u_n = \sin\frac{1}{n}$ & $v_n = \frac{1}{n}$, then

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1 \neq 0$$

$\therefore \Sigma u_n$ & Σv_n have same behaviour

W.k.T $\Sigma v_n = \Sigma \frac{1}{n}$ is dgt

$\therefore \Sigma u_n = \Sigma \sin\frac{1}{n}$ is dgt.

Ex. Test for cgt. of the series whose n^{th} term is $\{(n^3+1)^{1/3} - n\}$

Sol^n Let $u_n = (n^3+1)^{1/3} - n$

$$= n\left[\left(1+\frac{1}{n^3}\right) - 1\right]$$

$$= n\left[\left(1 + \frac{1}{3}\cdot\frac{1}{n^3} + \cdots\cdots\right) - 1\right]$$

$$= n\cdot\left[\frac{1}{3n^3}\right] = \frac{1}{3n^2}$$

Let $v_n = \frac{1}{n^2}$ then

$$\lim \frac{u_n}{v_n} = \lim \frac{n^2}{3n^2} = \frac{1}{3} \neq 0$$

$\therefore \Sigma u_n$ & Σv_n have same both behaviours

W.k.T $\Sigma v_n = \Sigma \frac{1}{n^2}$ is cgt

$\therefore \Sigma u_n$ is cgt.

Scanned with CamScanner

Ex. Test the cgt. of $\sum \frac{1}{n^{1+1/n}}$

Soln.

Let $u_n = \frac{1}{n^{1+1/n}}$, $v_n = \frac{1}{n}$, then

$$\lim \frac{u_n}{v_n} = \lim \frac{n}{n^{1+1/n}} = \lim \frac{n}{n \cdot n^{1/n}}$$

$$= \lim \frac{1}{n^{1/n}} = 1 \neq 0 \quad \left(\because \lim n^{1/n} = 1\right)$$

$\therefore \sum u_n$ & $\sum v_n$ have same behaviour

W.K.T $\sum v_n = \sum \frac{1}{n}$ is dgt

$\therefore \sum u_n$ is also dgt.

Ex: Test for cgt of the series whose general term is $\left[1 + \frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$

Soln.

Let $u_n = \left[1 + \frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$, then

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left[1 + \frac{1}{\sqrt{n}}\right]^{-n^{3/2} \cdot 1/n}$$

$$= \lim \left[1 + \frac{1}{\sqrt{n}}\right]^{-n^{1/2}}$$

$$= \lim_{n\to\infty} \frac{1}{\left[1 + \frac{1}{\sqrt{n}}\right]^{\sqrt{n}}}$$

$$= \frac{1}{e} \quad \left(\because \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e\right)$$

< 1 $\therefore$ given series is cgt (by C-R test)

Scanned with CamScanner

Ex. Test the cgt. of $\sum\left(\frac{n^2-1}{n^2+1}\right)x^n$, $x>0$

Soln: Let $u_n=\left(\frac{n^2-1}{n^2+1}\right)x^n$, then

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left[\frac{(n+1)^2-1}{(n+1)^2+1}\right]x^{n+1}\times\left(\frac{n^2+1}{n^2-1}\right)x^{-n}$$

$$=\lim_{n\to\infty}\frac{n^2}{n^2}\left[\frac{(1+1/n)^2-1/n^2}{(1+1/n)^2+1/n^2}\right]\times\frac{n^2(1+1/n^2)}{n^2(1-1/n^2)}\times x$$

$$=\left(\frac{1-0}{1+0}\right)\left(\frac{1+0}{1-0}\right)x=x$$

∴ By D'Alembert ratio test, given series $\sum u_n$ is cgt. if $x<1$, dgt. if $x>1$

The test fails when $x=1$

For $x=1$, $u_n=\frac{n^2-1}{n^2+1}$

$$\therefore n\to\infty\ u_n=\frac{1-1/n^2}{1+1/n^2}=1\neq 0$$

∴ $\sum u_n$ is dgt. for $x=1$

Hence $\sum u_n$ is cgt if $x<1$ & dgt if $x\geq 1$.

Scanned with CamScanner

Thm: 15 State & Prove Cauchy Integral test

Stat: If u is a non negative mono. decreasing integrable function such that $u(n) = u_n$, $\forall n \in N$ then P.T. the series $\sum u_n$ & $\int_1^\infty u(x)\,dx$ converge or diverge together.

Proof: As u is mono. decreasing

$\therefore u(n) \geq u(x) \geq u(n+1)$, whenever $n \leq x \leq n+1$

Also since u is non-ve & integrable

$$\therefore \int_n^{n+1} u(n)\,dx \geq \int_n^{n+1} u(x)\,dx \geq \int_n^{n+1} u(n+1)\,dx$$

$$\Rightarrow u(n)[x]_n^{n+1} \geq \int_n^{n+1} u(x)\,dx \geq u(n+1)[x]_n^{n+1}$$

$$\Rightarrow u(n) \geq \int_n^{n+1} u(x)\,dx \geq u(n+1) \quad \text{---(1)}$$

Let $S_n = \sum_{i=1}^{n} u_i$ & $I_n = \int_1^n u(x)\,dx$

Putting $n = 1, 2, 3, \ldots n-1$ in eqn (1),

$$\left.\begin{aligned} u_1 &\geq \int_1^2 u(x)\,dx \geq u_2 \\ u_2 &\geq \int_2^3 u(x)\,dx \geq u_3 \\ &\vdots \\ u_{n-1} &\geq \int_{n-1}^{n} u(x)\,dx \geq u_n \end{aligned}\right\} \text{---(2)}$$

Scanned with CamScanner

By adding all result of (2), we get

$$u_1+u_2+\dots+u_{n-1} \geq \int_1^2 u(x)\,dx + \int_2^3 u(x)\,dx + \dots + \int_{n-1}^{n} u(x)\,dx$$

$$\geq u_2+u_3+\dots+u_n$$

$$\Rightarrow S_n - u_n \geq I_n \geq S_n - u_1$$

$$\Rightarrow -u_n \geq I_n - S_n \geq -u_1$$

$$\Rightarrow 0 < u_n \leq S_n - I_n \leq u_1 \quad \forall n \quad \text{—(3)}$$

Consider seq. $\{S_n - I_n\}$, Then

$$(S_n - I_n) - (S_{n-1} - I_{n-1})$$

$$= S_n - S_{n-1} - (I_n - I_{n-1})$$

$$= u_n - \int_{n-1}^{n} u(x)\,dx \leq 0 \quad (\text{by (2)})$$

Thus $(S_n - I_n) \leq (S_{n-1} - I_{n-1})$, where $\forall n$, $n > n-1$

$\therefore$ seq. $\{S_n - I_n\}$ is mono. decreasing & bounded ($\because 0 \leq S_n - I_n \leq u_1$)

$\therefore \{S_n - I_n\}$ is cgt

Also $0 \leq S_n - I_n \leq u_1$

$\therefore 0 \leq \lim(S_n - I_n) \leq u_1$

$\Rightarrow 0 \leq \lim S_n - \lim I_n \leq u_1$

$\Rightarrow \lim I_n \leq \lim S_n$ &

$\lim S_n \leq u_1 + \lim I_n$

Scanned with CamScanner

$\Rightarrow$ seq. $\{S_n\}$ & $\{I_n\}$ converges or diverges together

Hence $\sum u_n$ & $\int_1^\infty u(x)$ cgt or dgt together.

Ex: Using Cauchy integral test, P.T. $\sum \frac{1}{n^p}$ converges if $p>1$ & diverges if $p \leq 1$.

Soln: Let $u_n = \frac{1}{n^p}$ & $u(n) = u_n$ $\forall n$

$\therefore u(x) = \frac{1}{x^p}$

Now for $x \geq 1$ & $p > 0$, $u(x)$ is +ve & mono. decreasing

$\therefore$ Cauchy integral test is applicable.

Case (i): If $p \neq 1$, then

$$I_n = \int_1^n u(x)\,dx = \int_1^n \frac{1}{x^p}\,dx = \int_1^n x^{-p}\,dx$$

$$= \left[\frac{x^{-p+1}}{-p+1}\right]_1^n$$

subcase-(I) If $p>1$ then $\Rightarrow p-1>0$

$$\therefore I_n = \frac{-1}{p-1}\left[\frac{1}{x^{p-1}}\right]_1^n$$

$$= \frac{1}{p-1}\left[1 - \frac{1}{n^{p-1}}\right]$$

Scanned with CamScanner

$\therefore \int_1^\infty u(x)\,dx = \int \lim_{n\to\infty} I_n$

$= \lim_{n\to\infty} \frac{1}{p-1}\left[1 - \frac{1}{n^{p-1}}\right]$

$= \frac{1}{p-1}\left[1 - 0\right] \quad \left(\because n\to\infty \Rightarrow \frac{1}{n^{p-1}} = 0\right)$

$= \frac{1}{p-1}$, which is finite

$\Rightarrow \int_1^\infty u(x)\,dx$ is cgt.

$\therefore \sum u_n$ is cgt $\left(\because$ by cauchy integral test$\right]$

Subcase (II) If $0 < p < 1$, then $1-p > 0$

$\therefore I_n = \frac{1}{1-p}\left\{x^{1-p}\right\}_1^n = \frac{1}{1-p}\left[n^{1-p} - 1\right]$

$\therefore \int_1^\infty u(x)\,dx = \lim I_n$

$= \lim \frac{1}{1-p}\left[n^{1-p} - 1\right]$

$= \infty \quad \left(\because \lim_{n\to\infty} n^{1-p} = \infty \atop (1-p>0)\right)$

$\therefore \int_1^\infty u(x)\,dx$ is dgt

$\therefore \sum u_n$ is dgt (by cauchy int. test)

Scanned with CamScanner

Case-(ii) If $p=1$, then

$$I_n = \int_1^n u(x)\,dx = \int_1^n \frac{1}{x}\,dx$$

$$= [\log x]_1^n = \log n - \log 1 = \log n$$

$$\therefore \int_1^\infty u(x)\,dx = \lim_{n\to\infty} I_n$$

$$= \lim_{n\to\infty} (\log n) = \infty$$

$\Rightarrow \int_1^\infty u(x)\,dx$ is dgt

$\therefore \sum u_n$ is dgt (by Cauchy integ. test)

By case (i) & (ii) result is proved

Ex: Using Cauchy integral test, discuss the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, \quad p>0$$

Soln: Let $u_n = \frac{1}{n(\log n)^p}$ & $u(n)=u_n$ $\forall n \geq 2$

Then

$$u(x) = \frac{1}{x(\log x)^p}, \quad x \geq 2$$

Thus $u(x)$ is +ve & mono. decreasing

$\therefore$ Cauchy integral test is applicable

$\therefore \sum_2^\infty u_n$ & $\int_2^\infty u(x)dx$ have same behaviour

Scanned with CamScanner

Case-(i) If $p \neq 0$, then

$$I_n = \int_2^n u(x)\,dx = \int_2^n \frac{1}{x(\log x)^p}\,dx$$

Let $\log x = t$ then $\frac{1}{x}dx = dt$

Also for $x=2$, $t=\log 2$

for $x=n$, $t=\log n$

$$\therefore I_n = \int_{\log 2}^{\log n} \frac{1}{t^p}\,dt = \left[\frac{t^{-p+1}}{-p+1}\right]_{\log 2}^{\log n}$$

$$I_n = \frac{1}{(1-p)}\left[t^{1-p}\right]_{\log 2}^{\log n}$$

Subcase-(I) If $p>1$, then $p-1>0$

$$\therefore I_n = \frac{-1}{p-1}\left[\frac{1}{t^{p-1}}\right]_{\log 2}^{\log n}$$

$$= -\frac{1}{p-1}\left[\frac{1}{(\log n)^{p-1}} - \frac{1}{(\log 2)^{p-1}}\right]$$

$$\therefore \int_2^\infty u(x)\,dx = n \to \infty\, I_n$$

$$= -\frac{1}{(p-1)}\left[0 - \frac{1}{(\log 2)^{p-1}}\right]$$

$$= \frac{1}{(p-1)(\log 2)^{p-1}}, \text{ which is finite}$$

$\Rightarrow \int_2^\infty u(x)\,dx$ is cgt

$\Rightarrow \sum_2^\infty u_n$ is cgt (by integral test)

Scanned with CamScanner

Subcase (II) If $p<1$, then $1-p>0$

$$\therefore I_n = \frac{1}{1-p}\left[(\log n)^{1-p} - (\log 2)^{1-p}\right]$$

$$\therefore \int_2^\infty u(x)\,dx = \lim I_n$$

$$= \lim\left[\frac{(\log n)^{1-p} - (\log 2)^{1-p}}{1-p}\right]$$

$$= \frac{\infty - (\log 2)^{1-p}}{1-p}$$

$$= \infty$$

$\Rightarrow \int_2^\infty u(x)\,dx$ is dgt

$\Rightarrow \sum_2^\infty u_n$ is dgt

Case-(ii) If $p=1$, then

$$I_n = \int_2^n u(x)\,dx = \int_2^n \frac{1}{x\log x}\,dx$$

Let $\log x = t$ then $\frac{1}{x}dx = dt$

for $x=2$, $t=\log 2$

$x=n$ $t=\log n$

$$\therefore I_n = \int_{\log 2}^{\log n} \frac{1}{t}\,dt = [\log t]_{\log 2}^{\log n}$$

$$\Rightarrow I_n = \log(\log n) - \log(\log 2)$$

$$\therefore \int_2^\infty u(x)\,dx = n\to\infty \lim I_n = \infty$$

Scanned with CamScanner

$\Rightarrow \int_2^\infty u(x)\,dx$ is dvt.

$\therefore \sum_2^\infty u_n$ is dgt (by integral test)

By case (i) & (ii) we say that result is proved.

i.e $\sum \frac{1}{n(\log n)^p}$ is cgt if $p>1$

& is dgt if $p \le 1$

Scanned with CamScanner

Defn: Alternating series :-

A series whose terms are alternatively +ve & -ve is called an alternating series

e.g $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots$

Theorem: 16 State & Prove Leibnitz Test for alternating series

Proof: statement: If alternating series $u_1-u_2+u_3-u_4+\cdots$, $u_n>0\ \forall n$ is such that

i) $u_{n+1}\leq u_n\ \forall n$ and

ii) $\lim u_n=0$ then P.T. the series is convergence.

Proof:- Let $S_n=u_1-u_2+u_3-u_4+\cdots+(-1)^{n-1}u_n$ be the n^{th} partial sum of $\sum(-1)^{n-1}u_n$.

For all $n\in N$

$$S_{2n+2}-S_{2n}=[u_1-u_2+u_3-\cdots+u_{2n+1}-u_{2n+2}]-[u_1-u_2+u_3-\cdots-u_{2n}]$$

$$=u_{2n+1}-u_{2n+2}\geq 0\ (\because u_n\geq u_{n+1})$$

$\Rightarrow S_{2n+2}\geq S_{2n}\ \forall n$

$\Rightarrow \{S_{2n}\}$ is mono. increasing seq.

Also $S_{2n}=u_1-u_2+u_3-\cdots+u_{2n-1}-u_{2n}$

$=u_1-(u_2-u_3)-(u_4-u_5)-\cdots$

$\cdots-(u_{2n-2}-u_{2n-1})-u_{2n}$

$<u_1$ ($\because$ each bracket on RHS is +ve $\because u_{n+1}\leq u_n$)

$\therefore$

Scanned with CamScanner

$\Rightarrow \{S_{2n}\}$ is bdd above

Thus $\{S_{2n}\}$ is mono. increasing & bdd above

$\therefore \{S_{2n}\}$ is cgt.

Let $\lim S_{2n} = S$

Now we p.T $\lim S_{2n+1} = S$

W.k.T $S_{2n+1} = a_1 - a_2 + a_3 - \cdots - a_{2n} + a_{2n+1}$

$\Rightarrow S_{2n+1} = S_{2n} + a_{2n+1}$

$\Rightarrow \lim S_{2n+1} = \lim S_{2n} + \lim a_{2n+1}$

$\Rightarrow = S + 0$

$= S \quad (\because \lim a_n = 0)$

Thus $\{S_{2n}\}$ & $\{S_{2n+1}\}$ both converges to the same limit S.

Now we p.T $\{S_n\}$ cgt to S.

For given $\varepsilon > 0$.

W.k.T $\lim S_{2n} = S$ & $\lim S_{2n+1} = S$

$\therefore \exists\, m_1, m_2 \in \mathbb{N}$ ∋

$|S_{2n} - S| < \varepsilon \quad \forall\, n \geq m_1$ —— (1)

$|S_{2n+1} - S| < \varepsilon \quad \forall\, n \geq m_2$ —— (2)

Let $m = \max\{2m_1, 2m_2+1\}$, then by (1) & (2) we say that

$|S_n - S| < \varepsilon \quad \forall\, n \geq m$

Thus $\{S_n\}$ cgt. to S

Hence given series $\sum (-1)^{n-1} a_n$ is cgt.

Scanned with CamScanner

Def 1: Absolutely convergent series:-

A series $\sum u_n$ is said to be absolutely cgt, if $\sum |u_n|$ is cgt.

Conditional cgt. series:-

A series which is cgt but not absolutely cgt is called a conditional cgt. series

Ex. P.T $1-\frac{1}{2^3}+\frac{1}{3^3}-\dots$ is absolutely cgt.

Soln $\sum |u_n| = 1+\frac{1}{2^3}+\frac{1}{3^3}+\dots = \sum \frac{1}{n^3}$

which is cgt. ($\because \sum \frac{1}{n^p}$ is cgt if $p>1$)

$\therefore$ Given series is absolutely cgt.

Ex. P.T $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$ is conditionally cgt.

Soln. For $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$

W.K.T $u_{n+1} \le u_n \ \forall n$ &

$\lim u_n = \lim (-1)^{n-1}\frac{1}{n} = 0$

$\therefore$ by Leibnitz test,

Given series is cgt.

Also $\sum |u_n| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$

is divergent ($\because \sum \frac{1}{n^p}$ is dgt if $p=1$)

Thus given series is conditionally cgt.

Scanned with CamScanner

Thm: 17 P.T every absolutely cgt series is cgt.

Proof: Let $\sum a_n$ be absolutely cgt then $\sum |a_n|$ is cgt

$\therefore$ For any $\varepsilon > 0$, by Cauchy general principle of convergence, $\exists\ m \in \mathbb{N} \ni$

$$|a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}| < \varepsilon \quad \forall\ n \geq m,\ p \geq 1$$

$$\Rightarrow |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}| < \varepsilon \quad \text{---(1)}$$

$$\Rightarrow |a_{n+1} + a_{n+2} + \cdots + a_{n+p}|$$

$$\leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}|$$

$$< \varepsilon \quad \forall\ n \geq m \ \&\ p \geq 1 \quad (\text{by } (1))$$

$\therefore$ By cauchy general principle we say that $\sum a_n$ is cgt.

Ex: Give an example of series $\sum a_n$ for which $\sum |a_n|$ is dgt but $\sum a_n$ is not dgt.

Soln Let $\sum \frac{(-1)^{n-1}}{n}$ $\left(= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right)$

then is is cgt.

but $\sum |a_n| = \sum \frac{1}{n}$ is dgt.

Scanned with CamScanner

Ex. S.T $\sum \frac{\sin nx}{n^2}$ is cgt. for any fixed value of x

Soln Let $\sum u_n = \sum \frac{\sin nx}{n^2}$ then

$u_n = \frac{\sin nx}{n^2}$ & $|u_n| = \frac{|\sin nx|}{n^2}$

Nw. let.T $\left|\frac{\sin nx}{n^2}\right| = \frac{|\sin nx|}{n^2} \leq \frac{1}{n^2} \quad \forall n$

$(\because |\sin nx| \leq 1)$

i.e $|u_n| \leq v_n \quad \forall n$

$\therefore$ By comparision Test

since $\sum v_n = \sum \frac{1}{n^2}$ is cgt

$\therefore \sum |u_n| = \sum \left|\frac{\sin nx}{n^2}\right|$ is cgt

$\therefore$ by above thm. $\sum u_n = \sum \frac{\sin nx}{n^2}$ is cgt.

Ex. P.T $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$ converges absolutely for all values of x

Soln Let $u_n = x^n/n!$ then

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \times \frac{n!}{|x|^n}$$

$$= \lim_{n\to\infty} \frac{|x|}{(n+1)}$$

$$= 0 < 1$$

Scanned with CamScanner

∴ By Ratio test, we say that
$\sum |u_n|$ is cgt

∴ $\sum u_n$ is absolutely cgt for all x

Ex. Remark: $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

Ex. S.T $\lim \frac{m(m-1)\cdots(m-n+1)}{(n-1)!} x^n = 0$
where $|x|<1$ & $m \in \mathbb{R}$

Soln. Consider the series $\sum u_n$, where
$u_n = \frac{m(m-1)\cdots(m-n+1)}{(n-1)!} x^n$ then

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \left| \frac{m(m-1)\cdots(m-n)}{n!} x^{n+1} \right| \times \left| \frac{(n-1)!}{m(m-1)\cdots(m-n+1)} \frac{1}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(m-n)}{n} \right| |x|$$

$$= n\to\infty \left| \frac{n(m/n - 1)}{n} \right| |x| = \left| \frac{(0-1)}{1} \right| |x| = |x|$$

$\Rightarrow \sum |u_n|$ is cgt if $|x|<1$

∴ $\sum u_n$ is absolutely cgt if $|x|<1$

$\Rightarrow \sum u_n$ is cgt for $|x|<1$

$\Rightarrow \lim u_n = 0$ for $|x|<1$

$(\because \sum u_n$ is cgt then $\lim u_n = 0)$

Scanned with CamScanner

Ex: Test the convergence of the following series

1) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\cdots\cdots$

Soln We can write given series as

$\left(\frac{3}{5}+\frac{3}{5^3}+\frac{3}{5^5}+\cdots\right)+\left(\frac{4}{5^2}+\frac{4}{5^4}+\frac{4}{5^6}+\cdots\right)$

$= \sum u_n + \sum v_n$

W.K.T $\sum u_n$ is geometric series with common ratio $r = 1/5^2 < 1$

$\therefore \sum u_n$ is cgt.

Also $\sum v_n$ is geometric series with ratio $r = 1/5^2 < 1$

$\therefore \sum v_n$ is cgt

$\therefore \sum u_n + \sum v_n$ is cgt

$\therefore$ given series is cgt.

2) $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\cdots\cdots$

Soln Here given series

$=\left(\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\cdots\right)+\left(\frac{1}{3^2}+\frac{1}{3^4}+\cdots\right)$

$= \sum u_n + \sum v_n$

W.K.T $\sum u_n$ is Geometric series with ratio $r = 1/2^2 < 1$

$\therefore \sum u_n$ is cgt.

Scanned with CamScanner

Also $\sum v_n$ is geometric series with ratio $r = 1/3^2 < 1$

$\therefore \sum v_n$ is cgt.

$\therefore \sum u_n + \sum v_n$ is cgt

$\therefore$ Given series is cgt.

3) $a+b+a^2+b^2+a^3+b^3+\cdots\cdots$

Soln Here given series

$= (a+a^2+a^3+\cdots) + (b+b^2+b^3+\cdots)$

$= \sum u_n + \sum v_n$

W.K.T $\sum u_n$ & $\sum v_n$ are geometric series with common ratio $r = a$ & $r = b$ resp.

$\therefore \sum u_n$ is cgt if $|a|<1$ &

$\sum v_n$ is cgt if $|b|<1$

$\therefore \sum u_n + \sum v_n$ is cgt if $|a|<1$ & $|b|<1$

$\therefore$ Given series is cgt if $|a|<1$ & $|b|<1$

4) $1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} + \cdots$

Soln $= 1 + \frac{1}{2^{4/3}} + \frac{1}{3^{4/3}} + \frac{1}{4^{4/3}} + \cdots$

$= \sum \frac{1}{n^{4/3}}$ where $p = \frac{4}{3} > 1$

$\therefore$ It is cgt.

Scanned with CamScanner

5) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots\cdots$

Soln Here $\sum u_n = \sum \frac{1}{n(n+1)}$

$\therefore u_n = \frac{1}{n(n+1)} = \frac{1}{n^2(1+1/n)}$

Let $v_n = \frac{1}{n^2}$ then

$\lim \frac{u_n}{v_n} = \lim \frac{n^2}{n^2(1+\frac{1}{n})} = 1 \neq 0$

$\therefore \sum u_n$ & $\sum v_n$ have same behaviour,

since $\sum v_n = \sum \frac{1}{n^2}$ is cgt. ($\because p=2>1$)

$\therefore \sum u_n$ is cgt.

6) $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \cdots$

Soln Here $u_n = \frac{1}{n(n+3)} = \frac{1}{n^2(1+\frac{3}{n})}$

Let $v_n = 1/n^2$, then

$\lim \frac{u_n}{v_n} = \lim \frac{n^2}{n^2(1+\frac{3}{n})} = 1 \neq 0$

$\therefore \sum u_n$ & $\sum v_n$ have same behaviour

since $\sum v_n = \sum \frac{1}{n^2}$ is cgt ($\because p=2>1$)

$\therefore \sum u_n$ is also cgt

7) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \cdots$

soln Here $u_n = \frac{2n-1}{n(n+1)(n+2)}$

$= \frac{(2-1/n)}{n^2(1+1/n)(1+2/n)}$

Scanned with CamScanner

Let $v_n = 1/n^2$ then

$$\lim_{n\to\infty} \frac{a_n}{v_n} = \lim \frac{2 - 1/n}{(1+1/n)(1+2/n)} = 2 \neq 0$$

$\therefore \Sigma a_n$ & Σv_n have same behaviour

since $\Sigma v_n = \Sigma 1/n^2$ is cgt. $(\because p = 2 > 1)$

$\therefore \Sigma a_n$ is also cgt.

8) $\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots$

Soln Here $a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}(1+\sqrt{1+1/n})}$

Let $v_n = 1/\sqrt{n}$ then

$$\lim_{n\to\infty} \frac{a_n}{v_n} = \lim \frac{\sqrt{n}}{\sqrt{n}(1+\sqrt{1+1/n})} = \frac{1}{2} \neq 0$$

$\therefore \Sigma a_n$ & Σv_n have same behaviours

since $\Sigma v_n = \Sigma 1/\sqrt{n} = \Sigma \frac{1}{n^{1/2}}$ is dgt

$(\because p = \frac{1}{2} < 1)$

$\therefore \Sigma a_n$ is also dgt.

9) $\frac{1}{a+b} + \frac{1}{a\cdot 2^2+b} + \frac{1}{a\cdot 3^2+b} + \cdots$

Soln Here $a_n = \frac{1}{an^2+b} = \frac{1}{n^2(a+b/n^2)}$

Let $v_n = 1/n^2$ then

$$\lim \frac{a_n}{v_n} = \lim \frac{n^2}{n^2(a+b/n^2)} = \frac{1}{a} \neq 0$$

$\therefore \Sigma a_n$ & Σv_n have same behaviour

Scanned with CamScanner

since $\sum v_n = \sum 1/n^2$ is cgt ($\because p=2>1$)

$\therefore \sum u_n$ is cgt.

10) $1 + \frac{1}{3^p} + \frac{1}{5^p} + \cdots$

Soln Here $u_n = \frac{1}{(2n-1)^p} = \frac{1}{n^p(2-1/n)^p}$

Let $v_n = 1/n^p$ then

$$\lim_{n\to\infty} \frac{u_n}{v_n} = n\to\infty \frac{n^p}{n^p(2-\frac{1}{n})^p} = \frac{1}{2^p} \neq 0$$

$\therefore \sum u_n$ & $\sum v_n$ have same behaviours

since $\sum v_n = \sum \frac{1}{n^p}$ is cgt. if $p>1$ & dgt if $p\leq 1$

$\therefore \sum u_n$ is cgt if $p>1$ & dgt. if $p\leq 1$.

11) $2 + \frac{3}{2^p} + \frac{4}{3^p} + \cdots$

Soln Here $u_n = \frac{n+1}{n^p} = \frac{1+1/n}{n^{p-1}}$

Let $v_n = \frac{1}{n^{p-1}}$ then

$$\lim \frac{u_n}{v_n} = \lim \frac{n^{p-1}(1+1/n)}{n^{p-1}} = 1 \neq 0$$

$\therefore \sum u_n$ & $\sum v_n$ have same behaviours

since $\sum v_n = \sum \frac{1}{n^{p-1}}$ is cgt. if $p-1>1$ & dgt if $p-1\leq 1$

$\therefore \sum u_n$ is cgt if $p>2$ & dgt. if $p\leq 2$.

Scanned with CamScanner

12) $2^p + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \cdots$

Soln Here $u_n = \frac{(n+1)^p}{n^q} = \frac{n^p (1+1/n)^p}{n^{q-p}}$

Let $v_n = \frac{1}{n^{q-p}}$

$$\lim \frac{u_n}{v_n} = \lim \frac{n^{q-p}(1+1/n)^p}{n^{q-p}} = 1 \neq 0$$

$\therefore \Sigma u_n$ & Σv_n have same behaviours

since $\Sigma v_n = \Sigma \frac{1}{n^{q-p}}$ is cgt if $q-p>1$

& dgt if $q-p \leq 1$

$\therefore \Sigma u_n$ is cgt if $q-p>1$ & dgt. if $q-p \leq 1$.

13) $\frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots$

Soln $u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{n^{n+1}(1+1/n)^{n+1}}$

Let $v_n = 1/n$ then $= \frac{1}{n(1+1/n)^{n+1}}$

$$\lim \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} \times \frac{1}{(1+1/n)} = \frac{1}{e} \neq 0$$

$\therefore \Sigma u_n$ & Σv_n have same behaviours

Since $\Sigma v_n = \Sigma 1/n$ is dgt ($\because p=1$)

$\therefore \Sigma u_n$ is also dgt.

Scanned with CamScanner

14) $(\sqrt{2}-1)+(\sqrt{5}-2)+(\sqrt{10}-3)+\cdots$

Soln $a_n = \sqrt{n^2+1}-n$

$$= \sqrt{n^2+1}-n \times \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$$

$$= \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{n\left(\sqrt{1+1/n^2}+1\right)}$$

Let $v_n = 1/n$ then

$$\lim \frac{a_n}{v_n} = \lim \frac{n}{n\left(\sqrt{1+1/n^2}+1\right)} = \frac{1}{2} \neq 0$$

$\therefore \sum a_n$ & $\sum v_n$ have same behaviour

Since $\sum v_n = \sum \frac{1}{n}$ is dgt ($\because p=1$)

$\therefore \sum a_n$ is also dgt.

15) $\sum \frac{1}{n} \sin \frac{1}{n}$

Soln Here $a_n = \frac{1}{n} \sin \frac{1}{n}$

Let $v_n = 1/n^2$, then

$$\lim \frac{a_n}{v_n} = \lim \frac{n^2 \sin 1/n}{n}$$

$$= \lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1 \neq 0$$

$\therefore \sum a_n$ & $\sum v_n$ have same behaviour

Since $\sum v_n = \sum 1/n^2$ is cgt. ($\because p=2>1$)

$\therefore \sum a_n$ is cgt.

Scanned with CamScanner

16) $\sum \cos 1/n$

Sol^n $u_n = \cos\frac{1}{n} = 1 - \frac{1}{2!n^2} + \frac{1}{4!n^4} - \frac{1}{6!n^6} + \cdots$

$\Rightarrow \lim u_n = \lim\left[1 - \frac{1}{2!n^2} + \frac{1}{4!n^4} - \cdots\right]$

$= 1 \neq 0$

$\therefore \sum u_n$ is not cgt.

$\left[\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right]$

17) $\sum \tan^{-1} 1/n$

Sol^n Let $u_n = \tan^{-1} 1/n$, $v_n = 1/n$

then $\lim \frac{u_n}{v_n} = \lim \frac{\tan^{-1} 1/n}{1/n} = 1 \neq 0$

$\therefore \sum u_n$ & $\sum v_n$ have same behaviour

Since $\sum v_n = \sum 1/n$ is dgt ($\because p = 1$)

$\therefore \sum u_n$ is dgt.

18) $\sum \cot^{-1} n^2$

Sol^n Here $u_n = \cot^{-1} n^2 = \tan^{-1}(1/n^2)$

Let $v_n = 1/n^2$ then $\lim \frac{u_n}{v_n} = 1 \neq 0$

& $\sum v_n = \sum 1/n^2$ is cgt ($\because p = 2 > 1$)

$\therefore \sum u_n$ is also cgt

Scanned with CamScanner

19) $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots$

Sol^n Here $u_n = \frac{n!}{n^n}$ $\therefore u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\therefore \lim \frac{u_{n+1}}{u_n} = \lim \frac{(n+1)!\, n^n}{(n+1)^{n+1}\, n!}$$

$$= \lim \frac{(n+1)!\, n^n}{(n+1)^n (n+1)\, n!}$$

$$= \lim \frac{n^n}{(n+1)^n} = \lim \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \lim \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

$\therefore$ By Ratio test, $\sum u_n$ is cgt.

20) $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots$

Sol^n Here $u_n = \frac{1}{2^{n-1}+1}$ $\therefore u_{n+1} = \frac{1}{2^{n+1}+1}$

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{2^{n-1}+1}{2^n+1}$$

$$= \lim \frac{2^{n-1}\left[1 + 1/2^{n-1}\right]}{2^n\left[1 + 1/2^n\right]} = \frac{1}{2} < 1$$

$\therefore$ By Ratio test, $\sum u_n$ is cgt.

Scanned with CamScanner

19) $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots$

Soln Here $u_n = \frac{n!}{n^n}$ $\therefore u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\therefore \lim \frac{u_{n+1}}{u_n} = \lim \frac{(n+1)!\, n^n}{(n+1)^{n+1}\, n!}$$

$$= \lim \frac{(n+1)!\, n^n}{(n+1)^n (n+1)\, n!}$$

$$= \lim \frac{n^n}{(n+1)^n} = \lim \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \lim \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is cgt.

20) $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots$

Soln Here $u_n = \frac{1}{2^{n-1}+1}$ $\therefore u_{n+1} = \frac{1}{2^{n}+1}$

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{2^{n-1}+1}{2^n+1}$$

$$= \lim \frac{2^{n-1}\left[1 + 1/2^{n-1}\right]}{2^n \left[1 + 1/2^n\right]} = \frac{1}{2} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is cgt.

Scanned with CamScanner

21) $\frac{2}{1^2+1} + \frac{2^2}{2^2+1} + \frac{2^3}{3^2+1} + \cdots$

Soln Here $u_n = \frac{2^n}{n^2+1}$ $\therefore u_{n+1} = \frac{2^{n+1}}{(n+1)^2+1}$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^2+1} \times \frac{n^2+1}{2^n}$$

$$= \lim_{n\to\infty} \frac{2 \cdot n^2(1+1/n^2)}{n^2\left[(1+1/n)^2 + 1/n^2\right]} = 2 > 1$$

$\therefore$ By ratio test, $\sum u_n$ is dgt.

22) $\frac{1}{3} + \frac{2}{9} + \frac{27}{27} + \frac{64}{81} + \cdots$

Here $u_n = n^3/3^n$ $\therefore u_{n+1} = (n+1)^3/3^{n+1}$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \times \frac{3^n}{n^3}$$

$$= \lim_{n\to\infty} \frac{(1+1/n)^3}{3} = \frac{1}{3} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is cgt.

23) $\frac{1^2 \cdot 2^2}{1} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \cdots$

Soln Here $u_n = \frac{n^2(n+1)^2}{n!}$

$$\therefore u_{n+1} = \frac{(n+1)^2(n+2)^2}{(n+1)!}$$

Scanned with CamScanner

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2(n+2)^2}{(n+1)!} \times \frac{n!}{n^2(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{n^2(1+2/n)^2}{(n+1)\,n^2}$$

$$= \lim_{n\to\infty} \frac{(1+2/n)^2}{n(1+1/n)} = 0 < 1$$

$\therefore$ By ratio test, $\sum u_n$ is cgt.

24) $\frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \cdots$

Sol^n $u_n = \frac{1\cdot 2\cdot 3 \cdots n}{3\cdot 5\cdot 7 \cdots (2n+1)} = \frac{n!}{3\cdot 5\cdot 7 \cdots (2n+1)}$

$$\therefore u_{n+1} = \frac{(n+1)!}{3\cdot 5\cdot 7 \cdots (2n+1)(2n+3)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim \frac{(n+1)}{(2n+3)} = \lim \frac{1+1/n}{2+3/n} = \frac{1}{2} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is cgt.

25) $\frac{5}{2+5} + \frac{5^2}{2^2+5} + \frac{5^3}{2^3+5} + \cdots$

Sol^n Here $u_n = \frac{5^n}{2^n+5}$ $\therefore u_{n+1} = \frac{5^{n+1}}{2^{n+1}+5}$

$$\lim \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{5^{n+1}}{2^{n+1}+5} \times \frac{2^n+5}{5^n}$$

Scanned with CamScanner

$$= \lim_{n\to\infty} \frac{5\cdot 2^n (1+5/2^n)}{2^{n+1}(1+5/2^{n+1})} = \frac{5}{2} > 1$$

$\therefore$ By ratio test, $\sum a_n$ is dgt.

26) $\sum \frac{2^{n-1}}{3^n+1}$

Sol^n $a_n = \frac{2^{n-1}}{3^n+1}$, $a_{n+1} = \frac{2^n}{3^{n+1}+1}$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{2^n}{3^{n+1}+1} \times \frac{3^n+1}{2^{n-1}}$$

$$= \lim_{n\to\infty} \frac{2}{3^n[3+1/3^n]} \times 3^n[1+1/3^n]$$

$$= \frac{2}{3} < 1 \quad \therefore \text{By ratio test } \sum a_n \text{ is cgt}$$

27) $\sum \left(\frac{n}{n+1}\right)^{n^2}$

Sol^n $a_n = \left(\frac{n}{n+1}\right)^{n^2}$ $\therefore (a_n)^{1/n} = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n}$

$$\therefore \lim_{n\to\infty} (a_n)^{1/n} = \lim\left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = e^{-1} = \frac{1}{e} < 1$$

$\therefore$ By root test, $\sum a_n$ is cgt.

28) $\sum \frac{x^n}{n^n}$

Sol^n $a_n = \frac{x^n}{n^n}$ $\therefore (a_n)^{1/n} = \frac{x}{n}$

Scanned with CamScanner

$\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\frac{x}{n} = 0 < 1$

$\therefore \sum a_n$ is cgt

29) $\sum_{n=2}^{\infty}\frac{1}{(\log n)^n}$

$a_n = \frac{1}{(\log n)^n} \quad \therefore (a_n)^{1/n} = \frac{1}{\log n}$

$\therefore n\to\infty (a_n)^{1/n} = n\to\infty \frac{1}{\log n} = \frac{1}{\infty} = 0 < 1$

$\therefore$ By root test, $\sum a_n$ is cgt.

30) $\sum\left[\frac{n+1}{3n}\right]^n$

Soln: $a_n = \left[\frac{n+1}{3n}\right]^n \quad \therefore (a_n)^{1/n} = \frac{n+1}{3n}$

$\therefore \lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\frac{n(1+1/n)}{3n} = \frac{1}{3} < 1$

$\therefore$ By root test, $\sum a_n$ is cgt.

31) $(2^2-2)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \cdots$

Soln $a_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$

$\therefore (a_n)^{1/n} = \left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$

$\therefore \lim (a_n)^{1/n} = \lim\left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$

$= (e-1)^{-1} = \frac{1}{e-1} < 1$

Scanned with CamScanner

$\left(\because e>2 \therefore e-1>1 \therefore \frac{1}{e-1}<1\right)$

$\therefore$ By root test, $\sum u_n$ is cgt.

32) $\sum \frac{(n-\log n)^n}{2^n n^n}$

Soln $u_n = \frac{(n-\log n)^n}{2^n \cdot n^n} \quad \therefore u_n^{1/n} = \frac{n-\log n}{2n}$

$\therefore \lim (u_n)^{1/n} = \lim \frac{1}{2}\left(1-\frac{\log n}{n}\right)$

$= \frac{1}{2}(1-0)$ $\left(\because n\to\infty \ \frac{\log n}{n}\ \left(\frac{\infty}{\infty}\ \text{form}\right)\right.$

$= \frac{1}{2}<1$ $\left.= n\to\infty \ \frac{1/n}{1} = 0\right.$

$\therefore$ By Cauchy root test, $\sum u_n$ is cgt.

33) $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\cdots$

Soln $u_n = \left(\frac{n+1}{n+2}\right)^n x^n \quad \therefore (u_n)^{1/n} = \left(\frac{n+1}{n+2}\right)x$

$\therefore \lim (u_n)^{1/n} = \lim \left(\frac{1+1/n}{1+2/n}\right)x = x$

$\therefore$ By Root test, $\sum u_n$ is cgt, if $x<1$ & dgt if $x>1$

For $x=1$, $n\to\infty \ u_n = n\to\infty \left(\frac{n+1}{n+2}\right)^n$

$= \lim \left(\frac{1+1/n}{1+2/n}\right)^n = \lim \frac{(1+1/n)^n}{[(1+2/n)^{n/2}]^2}$

$= \frac{e}{e^2} = \frac{1}{e} \neq 0$

$\therefore \sum u_n$ is dgt, if $x=1$

Scanned with CamScanner

34) $\sum \left(\frac{nx}{n+1}\right)^n$

Soln $a_n = \left(\frac{nx}{n+1}\right)^n \quad \therefore (a_n)^{1/n} = \frac{nx}{n+1}$

$\therefore \lim (a_n)^{1/n} = \lim \frac{nx}{n+1} = \lim \frac{x}{1+1/n} = x$

$\therefore$ By Cauchy root test, $\sum a_n$ is cgt if $x<1$ & dgt if $x>1$

If $x=1$, $a_n = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+1/n}\right)^n$

$\lim a_n = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} \neq 0$

$\therefore \sum a_n$ is dgt, if $x=1$

35) $\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Soln $a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}} \quad \therefore a_n^{1/n} = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{1/2}}$

$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$

$\therefore \sum a_n$ is cgt.

36) $\sum 5^{-n-(-1)^n}$

Soln $a_n = 5^{-n-(-1)^n} \quad \therefore (a_n)^{1/n} = 5^{-1-\frac{(-1)^n}{n}}$

$\therefore \lim (a_n)^{1/n} = \lim 5^{-1-\frac{(-1)^n}{n}} = 5^{-1-0} = \frac{1}{5} < 1$

$\therefore$ By root test, $\sum a_n$ is cgt.

36) $\frac{1}{3} + \frac{2^3}{3^2} + \frac{3^3}{3^3} + \frac{4^3}{3^4} + \cdots$

Scanned with CamScanner

Sol^n: $u_n = \frac{n^3}{3^n} \therefore (u_n)^{1/n} = \frac{n^{3/n}}{3}$

$\lim (u_n)^{1/n} = \lim \frac{n^{3/n}}{3} = \lim \frac{1}{3}\left(n^{1/n}\right)^3 = \frac{1}{3} < 1$

$\therefore \Sigma u_n$ is cgt. $\left(\because n \to \infty \; n^{1/n} = 1\right)$

37) $\sum_{n=2}^{\infty} \frac{1}{[\log(\log n)]^n}$

$u_n = \frac{1}{[\log(\log n)]^n} \therefore u_n^{1/n} = \frac{1}{\log(\log n)}$

$\therefore \lim (u_n)^{1/n} = \lim \frac{1}{\log(\log n)} = \frac{1}{\infty} = 0 < 1$

$\left(\because \lim_{n\to\infty} \log(\log n) = \infty\right)$

$\therefore$ By C-R test, Σu_n is cgt

38) $2x + \frac{3^2}{2^3}x^2 + \frac{4^3}{3^4}x^3 + \cdots$

Sol^n $u_n = \frac{(n+1)^n}{n^{n+1}} x^n \therefore (u_n)^{1/n} = \frac{n+1}{n^{1+1/n}} x$

$\lim u_n = \lim \frac{n(1+1/n)}{n \cdot n^{1/n}} x = \frac{(1+0)x}{1} = x$

$\left(\because n \to \infty \; n^{1/n} = 1\right)$

By C-R test, Σu_n is cgt if $x < 1$, dgt if $x > 1$

For $x = 1$,

$u_n = \frac{(n+1)^n}{n \cdot n^n} = \frac{n^n (1+1/n)^n}{n \cdot n^n}$

Let $v_n = \frac{1}{n}$, $\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim (1+1/n)^n$

$= e > 1$

& $\Sigma v_n = \Sigma \frac{1}{n}$ is dgt

$\therefore \Sigma u_n$ is dgt, if $x = 1$.

Scanned with CamScanner

Ex: Test the convergence of following series, by Cauchy integral test.

① $\sum \frac{1}{n^2+1}$

Soln Here $u_n = \frac{1}{n^2+1}$, Let $u_n = u(n)$. $\forall n$

$\therefore u(x) = \frac{1}{x^2+1}$

$\Rightarrow$ for $x \geq 1$, $u(x) > 0$ & mono. decreasing fn.

$\therefore$ By Cauchy integral test $\sum u_n$ & $\int_1^\infty u(x)dx$ have same behaviour

Now $\int_1^\infty u(x)\,dx = \int_1^\infty \frac{1}{x^2+1}\,dx = [\tan^{-1}x]$

$= n \to \infty \int_1^n \frac{1}{x^2+1}\,dx = n \to \infty \left[\tan^{-1}x\right]_1^n$

$= \lim [\tan^{-1} n - \tan^{-1} 1]$

$= \tan^{-1}\infty - \pi/4 = \pi/2 - \pi/4 = \pi/4 =$ finite

$\therefore \int_1^\infty u(x)\,dx$ is cgt $\therefore \sum u_n$ is cgt.

② $\sum \frac{1}{2n+3}$

Soln $u_n = \frac{1}{2n+3}$, Let $u_n = u(n)$ $\forall n$

$\therefore u(x) = \frac{1}{2x+3}$

Here for $x \geq 1$, $u(x) > 0$ & mono. decreasing fn. $\therefore$ By Cauchy integral test $\sum u_n$ & $\int_1^\infty u(x)\,dx$ have same behaviour

Scanned with CamScanner

Now $\int_1^\infty u(x)\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{2x+3}\,dx$

$= \lim \left[\frac{1}{2}\log(2x+3)\right]_1^n$

$= \lim_{n\to\infty} \frac{1}{2}\left[\log(2n+3) - \log 5\right]$

$= \frac{1}{2}[\infty - \log 5] = \infty$ not finite

$\Rightarrow \int_1^\infty u(x)\,dx$ is dgt

$\Rightarrow \sum u_n$ is dgt.

③ $\sum \frac{1}{n(n+1)}$

Sol^n $u_n = \frac{1}{n(n+1)}$, $u_n = u(n)$ $\forall n$

$\therefore u(x) = \frac{1}{x(x+1)}$

for $x \geq 1$, $u(x) > 0$ & mono. decreasing

$\therefore$ By Cauchy integral test $\sum u_n$ & $\int_1^\infty u(x)dx$ have same behaviour

$\therefore \int_1^\infty u(x)\,dx = \lim \int_1^n \frac{1}{x(x+1)}\,dx$

$= \lim_{n\to\infty} \int_1^n \left[\frac{1}{x} - \frac{1}{x+1}\right]dx$

$= n\to\infty \left[\log x - \log(x+1)\right]_1^n$

$= n\to\infty \log n - \log(n+1) - \log 1 + \log 2$

$= n\to\infty \log\left(\frac{n}{n+1}\right) + \log 2$

$= n\to\infty \log\left(\frac{1}{1+1/n}\right) + \log 2 =$

$= \log 1 + \log 2 = \log 2 =$ finite

$\therefore \int_1^\infty u(x)dx$ is cgt $\therefore \sum u_n$ is cgt

Scanned with CamScanner