S.Y.B.Sc. : Semester - III

US03CMTH21

Numerical Methods

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US03CMTH21- UNIT : II

1. Forward and Backward differences

Forward and Backward differences
If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y, then $y_1 - y_0,\ y_2 - y_1,\ \ldots, y_n - y_{n-1}$ are called differences of y.

The forward difference or first forward differnce is defined as

$$\Delta y_r = y_{r+1} - y_r, \quad r = 0, 1, 2, \ldots, n-1$$

where Δ is called forward difference operator.

The backward difference or first backward difference is defined as

$$\nabla y_r = y_r - y_{r-1}, \quad r =, 1, 2, \ldots, n$$

where ∇ is called backward difference operator.

2. Higher order differences

Higher order differences
If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y, then the differences of first forward differnces are called second forward differences and denoted using Δ^2. Therefore,

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

In general the n^{th} order forward difference is defined as Therefore,

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r$$

Similarly the differences of first backward differnces are called second backward differences and denoted using ∇^2. Therefore,

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$$

In general,

$$\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}$$

3. Central difference

Central difference
If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y, then the central difference operator , denoted by δ is defined by

$$\delta y_{r-\frac{1}{2}} = y_r - y_{r-1}$$

We note that $\delta y_{r-\frac{1}{2}} = \Delta y_{r-1} = \nabla y_r$

4. Mean operator

Mean operator
If $y_0, y_1, y_2, \dots, y_n$ denote a set of values of y, then the mean operator , denoted by μ is defined by

$$\mu y_r = \frac{1}{2}(y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}})$$

5. Shift operator

Shift operator
If $y_0, y_1, y_2, \dots, y_n$ denote a set of values of y, then the shift operator , denoted by E is defined by

$$Ey_r = y_{r+1}$$

In general,

$$E^m y_r = y_{r+m}$$

6. In usual notations prove the following :

-

[1] $\Delta = E - 1$

Proof:
We have,

$$\begin{aligned}\Delta y_n &= y_{n+1} - y_n \\ &= Ey_n - y_n \\ &= (E-1)y_n\end{aligned}$$

Hence, $\Delta \equiv E - 1$

[2] $\nabla = 1 - E^{-1}$

Proof:

We have,

$$\begin{aligned}\nabla y_n &= y_n - y_{n-1}\\ &= y_n - E^{-1}y_n\\ &= (1-E^{-1})y_n\end{aligned}$$

Hence, $\nabla \equiv 1 - E^{-1}$

[3] $\Delta - \nabla = \delta^2$

Proof:
We have,

$$\begin{aligned}\delta^2 y_n &= \delta(\delta y_n)\\ &= \delta\left(y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}\right)\\ &= \delta y_{n+\frac{1}{2}} - \delta y_{n-\frac{1}{2}}\\ &= \left[y_{(n+\frac{1}{2})+\frac{1}{2}} - y_{(n+\frac{1}{2})-\frac{1}{2}}\right] - \left[y_{(n-\frac{1}{2})+\frac{1}{2}} - y_{(n-\frac{1}{2})-\frac{1}{2}}\right]\\ &= (y_{n+1} - y_n) - (y_n - y_{n-1})\\ &= \Delta y_n - \nabla y_n\\ &= (\Delta - \nabla)y_n\end{aligned}$$

Hence, $\Delta - \nabla \equiv \delta^2$

[4] $\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$

Proof:
We have,

$$\begin{aligned}\mu y_n &= \frac{1}{2}\left(y_{n+\frac{1}{2}} + y_{n-\frac{1}{2}}\right)\\ &= \frac{1}{2}\left(E^{\frac{1}{2}}y_n + E^{-\frac{1}{2}}y_n\right)\\ &= \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)y_n\end{aligned}$$

$$\therefore \quad \mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$$

[5] $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

Proof:

We have,

$$\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) y_n = E^{\frac{1}{2}} y_n - E^{-\frac{1}{2}} y_n$$
$$= y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}$$
$$= \delta y_n$$

Hence, $E^{\frac{1}{2}} - E^{-12} \equiv \delta$

[6] $\Delta - \nabla = \Delta\nabla$

Proof:
We have,

$$(\Delta\nabla)y_n = \Delta(\nabla y_n)$$
$$= \Delta(y_n - y_{n-1})$$
$$= \Delta y_n - \Delta y_{n-1}$$
$$= \Delta y_n - (y_n - y_{n-1})$$
$$= \Delta y_n - \nabla y_n$$
$$= (\Delta - \nabla)y_n$$

Hence, $\Delta - \nabla \equiv \Delta\nabla$

[7] $(1 + \Delta)(1 - \nabla) = 1$

Proof:
We have,

$$(1 + \Delta)(1 - \nabla)y_n = (1 + \Delta)[(1 - \nabla)y_n]$$
$$= (1 + \Delta)[y_n - \nabla y_n]$$
$$= (1 + \Delta)[y_n - (y_n - y_{n-1})]$$
$$= (1 + \Delta)y_{n-1}$$
$$= y_{n-1} + \Delta y_{n-1}$$
$$= y_{n-1} + (y_n - y_{n-1})$$
$$= y_n$$

Hence, $(1 + \Delta)(1 - \nabla) \equiv 1$

[8] $\Delta = E\nabla$

Proof:

We have,

$$\begin{aligned}(E\nabla)y_n &= E(\nabla y_n)\\ &= E(y_n - y_{n-1})\\ &= Ey_n - Ey_{n-1}\\ &= y_{n+1} - y_{(n-1)+1}\\ &= y_{n+1} - y_n\\ &= \Delta y_n\end{aligned}$$

Hence, $\Delta \equiv E\nabla$

[9] $E^{-1} \equiv 1 - \nabla$

Proof:
We have,

$$\begin{aligned}(1-\nabla)y_n &= (1-\nabla)y_n\\ &= y_n - \nabla y_n\\ &= y_n - (y_n - y_{n-1})\\ &= y_{n-1}\\ &= E^{-1}y_n\end{aligned}$$

Hence, $E^{-1} \equiv 1 - \nabla$

[10] $\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)(1+\Delta)^{\frac{1}{2}} = 2 + \Delta$

Proof:
First we show that $\Delta + 1 \equiv E$
We have,

$$\begin{aligned}(\Delta+1)y_n &= \Delta y_n + y_n\\ &= (y_{n+1} - y_n) + y_n\\ &= y_{n+1}\\ &= Ey_n\end{aligned}$$

Hence, $\Delta + 1 \equiv E$

Now,

$$\begin{aligned}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)(1+\Delta)^{\frac{1}{2}} &\equiv \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)E^{\frac{1}{2}}\\ &\equiv (E^{\frac{1}{2}+\frac{1}{2}} + E^{\frac{1}{2}-\frac{1}{2}})\\ &\equiv E + 1\\ &\equiv (\Delta+1)+1\\ &\equiv 2+\Delta\end{aligned}$$

Hence, $\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)(1+\Delta)^{\frac{1}{2}} \equiv 2+\Delta$

[11] $\mu=\sqrt{1+\frac{1}{4}\delta^2}$

Proof:

We have $\delta y_n=\left(y_{n+\frac{1}{2}}-y_{n+\frac{1}{2}}\right)=\left(E^{\frac{1}{2}}y_n-E^{-\frac{1}{2}}\right)y_n=\left(E^{\frac{1}{2}}-E^{-\frac{1}{2}}\right)y_n$

Hence, $\delta \equiv E^{\frac{1}{2}}-E^{-\frac{1}{2}}$

Also $\mu y_n=\frac{1}{2}\left(y_{n+\frac{1}{2}}+y_{n+\frac{1}{2}}\right)=\frac{1}{2}\left(E^{\frac{1}{2}}y_n+E^{-\frac{1}{2}}\right)y_n=\frac{1}{2}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)y_n$

Hence, $\mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)$

Now,

$$
\begin{aligned}
1+\frac{1}{4}\delta^2 &\equiv 1+\frac{1}{4}\left(E^{\frac{1}{2}}-E^{-\frac{1}{2}}\right)^2\\
&\equiv \frac{1}{4}\left(4+E-2+E^{-1}\right)\\
&\equiv \frac{1}{4}\left(E+2+E^{-1}\right)\\
&\equiv \frac{1}{4}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)^2\\
&\equiv \left[\frac{1}{2}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)\right]^2\\
&\equiv \mu^2
\end{aligned}
$$

Hence, $\mu^2=1+\frac{1}{4}\delta^2$

[12] $\Delta=\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{\delta^2}{4}}$

Proof:

We have $\delta y_n=\left(y_{n+\frac{1}{2}}-y_{n+\frac{1}{2}}\right)=\left(E^{\frac{1}{2}}y_n-E^{-\frac{1}{2}}\right)y_n=\left(E^{\frac{1}{2}}-E^{-\frac{1}{2}}\right)y_n$

Hence, $\delta \equiv E^{\frac{1}{2}}-E^{-\frac{1}{2}}$ - - - - (1)

Also,

$$
\begin{aligned}
\Delta y_n &= y_{n+1}-y_n\\
&= Ey_n-y_n\\
&= (E-1)y_n
\end{aligned}
$$

Hence, $\Delta \equiv E - 1$ - - - - (2) Now,

$$\begin{aligned}
\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}} &\equiv \frac{1}{2}\delta\left[\delta + \sqrt{4+\delta^2}\right] \\
&\equiv \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) + \sqrt{4 + \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2}\right] \\
&\equiv \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) + \sqrt{4 + E - 2 + E^{-1}}\right] \\
&\equiv \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) + \sqrt{E + 2 + E^{-1}}\right] \\
&\equiv \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) + \sqrt{\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)^2}\right] \\
&\equiv \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) + \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)\right] \\
&\equiv \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)(2E^{\frac{1}{2}}) \\
&\equiv E - 1 \\
&\equiv \Delta
\end{aligned}$$

Hence, $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$

[13] $e^{hD} = E$

Proof:

First let us define the differential operator by $Dy(x) = \frac{dy(x)}{dx}, n \in N$

Now, by Taylor's series, we have

$$y(x+h) = y(x) + h\frac{dy(x)}{dx} + \frac{h^2}{2!}\frac{d^2y(x)}{dx^2} + \frac{h^3}{3!}\frac{d^3y(x)}{dx^3} + \dots$$

$$= y(x) + hDy(x) + \frac{h^2}{2!}D^3y(x) + \frac{h^3}{3!}D^3y(x) + \dots$$

$$= \left[1 + hD + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \dots\right]y(x)$$

$$Ey(x) = e^{hD}y(x) \qquad \text{for } y(x+h) = Ey(x)$$

Hence $E \equiv e^{hD}$

7. **Locate and correct error in the following table of values**

x	2.5	3.0	3.5	4.0	4.5	5.0	5.5
y	4.32	4.83	5.27	5.47	6.26	6.79	7.23

Answer:
We have to locate an error in the following data

X	2.5	3	3.5	4	4.5	5	5.5
Y	4.32	4.83	5.27	5.47	6.26	6.79	7.23

Following is the diffrence table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$	$\Delta^5 Y$	$\Delta^6 Y$
2.5	4.32						
		0.51					
3	4.83		-0.07				
		0.44		-0.17			
3.5	5.27		-0.24		1		
		0.2		0.83		-2.68	
4	5.47		0.59		-1.68		5.38
		0.79		-0.85		2.7	
4.5	6.26		-0.26		1.02		
		0.53		0.17			
5	6.79		-0.09				
		0.44					
5.5	7.23						

Step - 1 : First we find a column such that the differences in the column have alternating signs and the next column has any two consecutive differences nearly same. We find that in above table this column is at $\Delta^4 Y$ as all the diffences in the column have alternating signs and the next column corresponding to $\Delta^5 Y$ has two consecutive differences -2.68 and 2.7 with nearly same absolute values.

Step - 2 : Now in the next column corresponding to $\Delta^6 Y$ we locate the differnce 5.38 of -2.68 and 2.7.

Step - 3 : We conclude that there is an error in $y = 5.47$ which directly corresponds to difference 5.38 in $\Delta^6 Y$ column.

Step - 4 : Now we find the central coefficient in the expansion of $(1-\epsilon)^6$ which is

$$\binom{6}{3} = \frac{6\times5\times4}{3\times2\times1} = 20$$

Step - 5 : We find the error by $\epsilon = \dfrac{5.38}{20} = 0.269 \approx 0.27$

(approximation is done as given values of y are with two decimal places.)

Step - 6 : Error correction, $5.47 + 0.27 = 5.74$

Thus, actual value of y is 5.74 instead of 5.47.

8. Locate and correct error in the following table of values

x	1	2	3	4	5	6	7	8
y	3010	3424	3802	4105	4472	4771	5051	5315

Answer:
We have to locate an error in the following data

X	1	2	3	4	5	6	7	8
Y	3010	3424	3802	4105	4472	4771	5051	5315

Following is the diffrence table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$	$\Delta^5 Y$	$\Delta^6 Y$	$\Delta^7 Y$
1	3010							
		414						
2	3424		-36					
		378		-39				
3	3802		-75		178			
		303		139		-449		
4	4105		64		-271		901	
		367		-132		452		-1580
5	4472		-68		181		-679	
		299		49		-227		
6	4771		-19		-46			
		280		3				
7	5051		-16					
		264						
8	5315							

Step - 1 : First we find a column such that the differences in the column have alternating signs and the next column has any two consecutive differences nearly same. We find that in above table this column is at $\Delta^4 Y$ as all the diffences in the column have alternating signs and the next column corresponding to $\Delta^5 Y$ has two consecutive differences -449 and 452 with nearly same absolute values.

Step - 2 : Now in the next column corresponding to $\Delta^6 Y$ we locate the differnce 901 of -449 and 452.

Step - 3 : We conclude that there is an error in $y = 4105$ which directly corresponds to difference 901 in $\Delta^6 Y$ column.

Step - 4 : Now we find the central coefficient in the expansion of $(1-\epsilon)^6$ which is

$\binom{6}{3} = \frac{6\times5\times4}{3\times2\times1} = 20$

Step - 5 : We find the error by $\epsilon = \dfrac{901}{20} = 45.05 \approx 45$

(approximation is done as given values of y are integers.)

Step - 6 : Error correction, $4105 + 45 = 4150$

Thus, actual value of y is 4150 instead of 4105.

9. Derive Newton's Forward Difference interpolation formula for equally spaced values of arguments.

Answer:
Suppose $(x_0, y_0), (x_1, y_1), (x_2, y_2) \ldots, (x_n, y_n)$ are $n+1$ tabulated values of x and y and it is required to find a polynomial function $y_n(x)$ of n^{th} degree such that

$$y_n(x_r) = y(x_r), \qquad r = 0, 1, 2, \ldots, n \quad \text{- - - (1)}$$

Let us assume that the tabulated values $x_0, x_1, x_2, \ldots, x_n$ are equally spaced at a distance h. Therefore,

$$x_r = x_0 + rh, \quad r = 1, 2, \ldots, n$$

Suppose, the n^{th} degree polynomial $y_n(x)$ can be written as

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \cdots + a_n(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1}) \quad \text{- - - (2)}$$

where the coefficients a_r are to be determined subject to (1).

Now,
$y_n(x_0) = y(x_0) \Rightarrow a_0 = y_0$
Also,

$$\begin{aligned} y_n(x_1) = y(x_1) &\Rightarrow a_0 + a_1(x_1 - x_0) = y_1 \\ &\Rightarrow y_0 + a_1(x_1 - x_0) = y_1 \\ &\Rightarrow a_1 = \frac{y_1 - y_0}{x_1 - x_0} \\ &\Rightarrow a_1 = \frac{\Delta y_0}{h} \end{aligned}$$

Similarly using $y_n(x_r) = y(x_r)$ we get,

$$a_2 = \frac{\Delta^2 y_0}{2!h^2}, \; a_3 = \frac{\Delta^3 y_0}{3!h^3}, \ldots a_n = \frac{\Delta^n y_0}{n!h^n}$$

Also for some p, if we assume $x = x_0 + ph$ then we have,

$$x - x_r = (x_0 + ph) - (x_0 + rh) = (p - r)h$$

Substitutin in (2) we get

$$y_n(x) = y_0 + \frac{\Delta y_0}{h}(ph) + \frac{\Delta^2 y_0}{2!h^2}(p(p-1)h^2 + \cdots + \frac{\Delta^n y_0}{n!h^n}(p(p-1)\ldots(p-n+1)h^n$$

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \frac{p(p-1)\ldots(p-n+1)}{n!}\Delta^n y_0$$

which is Newton's forward difference interpolation formula and it is useful for interpolation near begining of the set of tabular values.

10. Derive Newton's Backward Difference interpolation formula for equally spaced values of argument

Answer:

Suppose $(x_0, y_0), (x_1, y_1), (x_2, y_2) \ldots, (x_n, y_n)$ are $n+1$ tabulated values of x and y and it is required to find a polynomial function $y_n(x)$ of n^{th} degree such that

$$y_n(x_r) = y(x_r), \qquad r = 0, 1, 2, \ldots, n \quad \text{--- (1)}$$

Let us assume that the tabulated values $x_0, x_1, x_2, \ldots, x_n$ are equally spaced at a distance h. Therefore,

$$x_{n-r} = x_n - rh, \quad r = 1, 2, \ldots, n$$

Suppose, the n^{th} degree polynomial $y_n(x)$ can be written as

$$\begin{aligned} y_n(x) = a_0 + a_1(x - x_n) \quad & + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \\ & \cdots + a_n(x - x_n)(x - x_{n-1})(x - x_{n-2}) \ldots (x - x_1) \quad \text{--- (2)} \end{aligned}$$

where the coefficients a_r are to be determined subject to (1).

Now,
$y_n(x_n) = y(x_n) \Rightarrow a_0 = y_n$
Also,

$$\begin{aligned} y_n(x_{n-1}) = y(x_{n-1}) & \Rightarrow a_0 + a_1(x_{n-1} - x_n) = y_{n-1} \\ & \Rightarrow y_n + a_1(x_{n-1} - x_n) = y_{n-1} \\ & \Rightarrow a_1 = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} \\ & \Rightarrow a_1 = \frac{\nabla y_n}{h} \end{aligned}$$

Similarly using $y_n(x_r) = y(x_r)$ we get,

$$a_2 = \frac{\nabla^2 y_n}{2!h^2}, \; a_3 = \frac{\nabla^3 y_n}{3!h^3}, \ldots, \; a_n = \frac{\nabla^n y_n}{n!h^n}$$

Also for some p, if we assume $x = x_n + ph$ then we have,

$$x - x_{n-r} = (x_n + ph) - (x_n - rh) = (p + r)h$$

Substituting in (2) we get

$$\begin{aligned} y_n(x) =& y_n + \frac{\nabla y_n}{h}(ph) + \frac{\nabla^2 y_n}{2!h^2}(p(p+1)h^2) + \cdots + \frac{\nabla^n y_n}{n!h^n}(p(p+1)\ldots(p+n-1)h^n) \\ y_n(x) =& y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots + \frac{p(p+1)\ldots(p+n-1)}{n!}\nabla^n y_n \end{aligned}$$

which is Newton's backward difference interpolation formula and it is useful for interpolation near end of the set of tabular values.

11. For the following data the Forward and Backward Difference Tables are given next.

X	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
Y	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$	$\Delta^5 Y$	$\Delta^6 Y$	$\Delta^7 Y$	$\Delta^8 Y$	$\Delta^9 Y$
x_0	y_0									
		Δy_0								
x_1	y_1		$\Delta^2 y_0$							
		Δy_1		$\Delta^3 y_0$						
x_2	y_2		$\Delta^2 y_0$		$\Delta^4 y_0$					
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$				
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$			
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$		$\Delta^7 y_0$		
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		$\Delta^6 y_1$		$\Delta^8 y_0$	
		Δy_4		$\Delta^3 y_3$		$\Delta^5 y_2$		$\Delta^7 y_2$		$\Delta^9 y_0$
x_5	y_5		$\Delta^2 y_4$		$\Delta^4 y_3$		$\Delta^6 y_2$		$\Delta^8 y_0$	
		Δy_5		$\Delta^3 y_4$		$\Delta^5 y_3$		$\Delta^7 y_3$		
x_6	y_6		$\Delta^2 y_5$		$\Delta^4 y_4$		$\Delta^6 y_2$			
		Δy_6		$\Delta^3 y_5$		$\Delta^5 y_4$				
x_7	y_7		$\Delta^2 y_6$		$\Delta^4 y_5$					
		Δy_7		$\Delta^3 y_6$						
x_8	y_8		$\Delta^2 y_7$							
		Δy_8								
x_9	y_9									

TABLE 1. Forward Differences

X	Y	∇Y	$\nabla^2 Y$	$\nabla^3 Y$	$\nabla^4 Y$	$\nabla^5 Y$	$\nabla^6 Y$	$\nabla^7 Y$	$\nabla^8 Y$	$\nabla^9 Y$
x_0	y_0									
		∇y_1								
x_1	y_1		$\nabla^2 y_2$							
		∇y_2		$\nabla^3 y_3$						
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$					
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$				
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$		$\nabla^6 y_6$			
		∇y_4		$\nabla^3 y_5$		$\nabla^5 y_6$		$\nabla^7 y_7$		
x_4	y_4		$\nabla^2 y_5$		$\nabla^4 y_6$		$\nabla^6 y_7$		$\nabla^8 y_8$	
		∇y_5		$\nabla^3 y_6$		$\nabla^5 y_7$		$\nabla^7 y_8$		$\nabla^9 y_9$
x_5	y_5		$\nabla^2 y_6$		$\nabla^4 y_7$		$\nabla^6 y_8$		$\nabla^8 y_9$	
		∇y_6		$\nabla^3 y_7$		$\nabla^5 y_8$		$\nabla^7 y_9$		
x_6	y_6		$\nabla^2 y_7$		$\nabla^4 y_8$		$\nabla^6 y_9$			
		∇y_7		$\nabla^3 y_8$		$\nabla^5 y_9$				
x_7	y_7		$\nabla^2 y_8$		$\nabla^4 y_9$					
		∇y_8		$\nabla^3 y_9$						
x_8	y_8		$\nabla^2 y_9$							
		Δy_9								
x_9	y_9									

TABLE 2. Backward Differences

12. **For the following data a Difference Table is obtained and the differences used in Newoton's Forward and Backward Interpolation formulae are indicated.**

X	5	10	15	20	25	30	35
Y	10	12	15	20	28	37	45

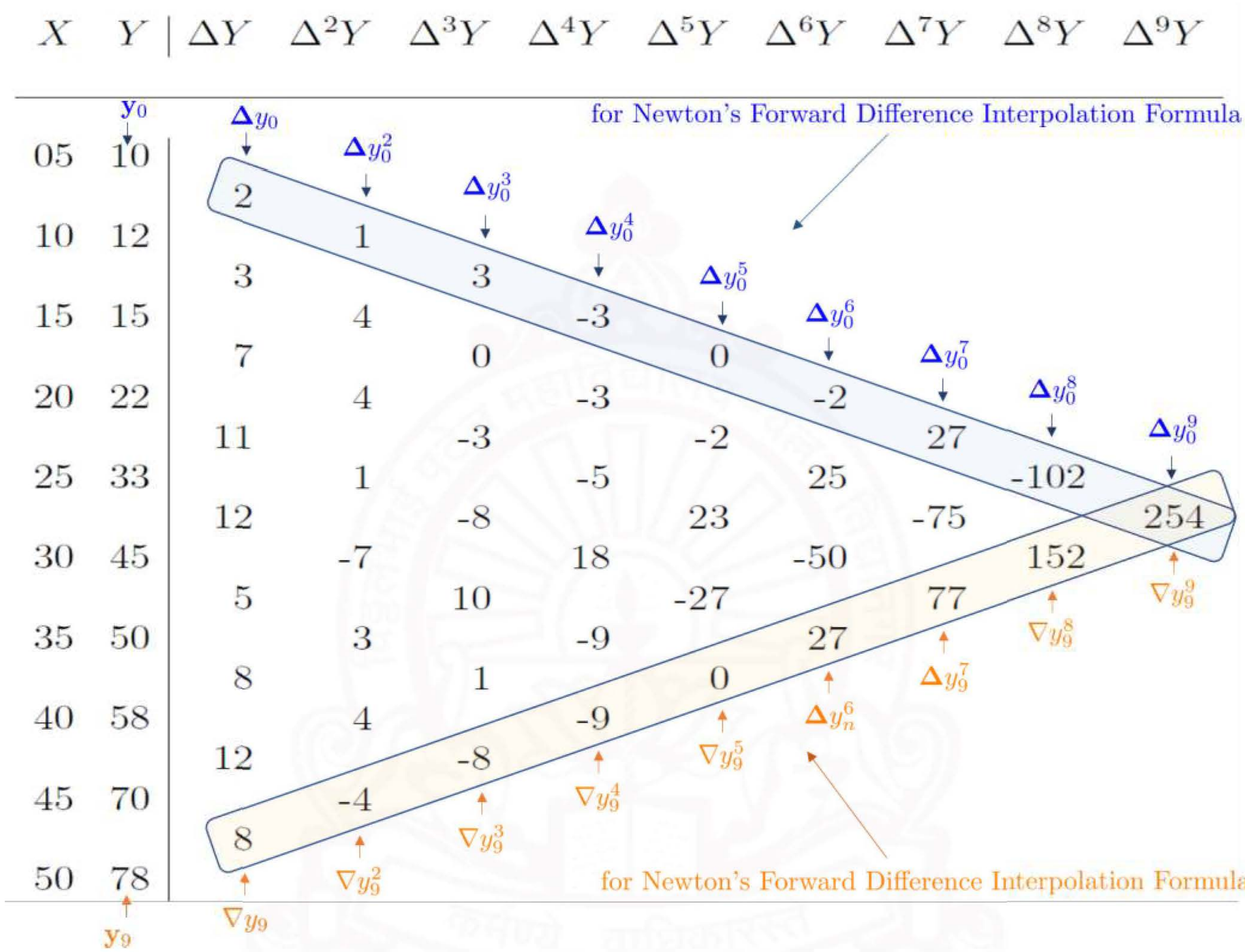

FIGURE 1. Difference Table

13. **The populations of a town were as under**

Year(x)	1891	1901	1911	1921	1931
Population (in thousand)	46	66	81	93	101

Estimate the population for the year 1895 and 1925

Answer:
For the given data

X	1891	1901	1911	1921	1931
Y	46	66	81	93	101

We have to find $y(1895)$

Following is the diffrence table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Here, we have h = 1901 - 1891 = 10

As, x = 1895 is near $x_0 = 1891$, we shall use Newton's Forward Difference Interpolation formula

Let $x = x_0 + ph$

Therefore $p = \dfrac{1895 - 1891}{10} = 0.4$

Substituting for p and differences in the **Newton's Forward Difference Interpolation formula**

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0 + \ldots$$

we get $y(1895) = y_p = 54.8528$

Again As, $x = 1925$ is near $x_n = 1931$, we shall use Newton's Backward Difference Interpolation formula

Let $x = x_n + ph$

Therefore $p = \dfrac{1925 - 1931}{10} = -0.6$

Substituting for p and differences in the **Newton's Backward Difference Interpolation formula**

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n + \ldots$$

we get $y(1925) = y_p = 96.8368$

14. Derive Gauss's Forward interpolation formula for equally spaced values of argument

Answer:
Suppose $\ldots,(x_{-3},y_{-3}),(x_{-2},y_{-2}),(x_{-1},y_{-1}),(x_0,y_0),(x_1,y_1),(x_2,y_2),(x_3,y_3),\ldots$ are some tabulated values of x and y such that (x_0,y_0) is the central data.

For a given x near x_0 suppose,

$$x = x_0 + ph$$

and correponding value of y is y_p

Suppose y_p can be expressed using the forward differences near the central observation as

$$y_p = y_0 + G_1\Delta y_0 + G_2\Delta^2 y_{-1} + G_3\Delta^3 y_{-1} + G_4\Delta^4 y_{-2} + G_5\Delta^5 y_{-2} + \ldots$$

where $G_1, G_2, \ldots$ are to be determined.

Now,

$$\begin{aligned} y_p &= E^p y_0 \\ &= (1+\Delta)^p y_0 \\ &= 1 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \end{aligned}$$

$$\therefore y_p = 1 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Also,

$$\begin{aligned} \Delta^2 y_{-1} &= \Delta^2\left(E^{-1}\right) y_0 \\ &= \Delta^2(1+\Delta)^{-1} y_0 \\ &= \Delta^2\left[1 - \Delta + \Delta^2 - \Delta^3 + \ldots\right] y_0 \\ &= \Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \ldots \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta^3 y_{-1} &= \Delta\left[\Delta^2 y_{-1}\right] \\ &= \Delta\left[\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \ldots\right] \\ &= \Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0 + \ldots \end{aligned}$$

Also,

$$\begin{aligned} \Delta^4 y_{-2} &= \Delta^4\left(E^{-2}\right) y_0 \\ &= \Delta^4(1+\Delta)^{-2} y_0 \\ &= \Delta^4\left[1 - 2\Delta + 3\Delta^2 - 4\Delta^3 + \ldots\right] y_0 \\ &= \Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \ldots \end{aligned}$$

Substituting on the L.H.S. and R.H.S. of (??) we get

$$\begin{aligned}1 + p\Delta y_0 + & \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \\ & + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots \\ = y_0 + & G_1\Delta y_0 + G_2\left[\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \dots\right] \\ & + G_3\left[\Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0 + \dots\right] \\ & + G_4\left[\Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \dots\right] \\ & + \dots\end{aligned}$$

Equating the coefficients of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \dots$ we get,

$$\begin{aligned}G_1 &= p, \\ G_2 &= \frac{p(p-1)}{2!}, \\ G_3 &= \frac{(p+1)p(p-1)}{3!}, \\ G_4 &= \frac{(p+1)p(p-1)(p-2)}{4!}, \\ G_5 &= \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}, \\ &\vdots\end{aligned}$$

Substituting in (??) we obtain the Gauss's Forward Interpolation formula given below

$$\begin{aligned}y_p = y_0 + p\Delta y_0 + & \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} \\ & + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} \\ & + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2} \\ & + \dots\end{aligned}$$

15. **For the following data the difference table assuming x_0 at the center is given in which the terms in blue color are used in Gauss's Forward Interpolation formula.**

X	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_9 x_5
Y	y_{-5}	y_{-4}	y_{-3}	y_{-2}	y_{-1}	y_0	y_1	y_2	y_3	y_9 x_5

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$	$\Delta^5 Y$	$\Delta^6 Y$	$\Delta^7 Y$	$\Delta^8 Y$	$\Delta^9 Y$	$\Delta^{10} Y$
x_{-5}	y_{-5}										
		Δy_{-5}									
x_{-4}	y_{-4}		$\Delta^2 y_{-5}$								
		Δy_{-4}		$\Delta^3 y_{-5}$							
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$		$\Delta^4 y_{-5}$						
		Δy_{-3}		$\Delta^3 y_{-4}$		$\Delta^5 y_{-5}$					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$		$\Delta^6 y_{-5}$				
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$		$\Delta^7 y_{-5}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		$\Delta^6 y_{-4}$		$\Delta^8 y_{-5}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$		$\Delta^7 y_{-4}$		$\Delta^9 y_{-5}$	
x_0	$\mathbf{y_0}$		$\mathbf{\Delta^2 y_{-1}}$		$\mathbf{\Delta^4 y_{-2}}$		$\mathbf{\Delta^6 y_{-3}}$		$\mathbf{\Delta^8 y_{-4}}$		$\mathbf{\Delta^{10} y_{-5}}$
		$\mathbf{\Delta y_0}$		$\mathbf{\Delta^3 y_{-1}}$		$\mathbf{\Delta^5 y_{-2}}$		$\mathbf{\Delta^7 y_{-3}}$		$\mathbf{\Delta^9 y_{-4}}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$		$\Delta^8 y_{-3}$		
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$		$\Delta^7 y_{-2}$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		$\Delta^6 y_{-1}$				
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$					
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$						
		Δy_3		$\Delta^3 y_2$							
x_4	y_4		$\Delta^2 y_3$								
		Δy_4									
x_5	y_5										

16. Derive Gauss's Backward interpolation formula for equally spaced values of argument

Answer:

Suppose $\ldots, (x_{-3}, y_{-3}), (x_{-2}, y_{-2}), (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ are some tabulated values of x and y such that (x_0, y_0) is the central data.

For a given x near x_0 suppose,

$$x = x_0 + ph$$

and correponding value of y is y_p

Suppose y_p can be expressed using the forward differences near the central observation as

$$y_p = y_0 + G_1'\Delta y_{-1} + G_2'\Delta^2 y_{-1} + G_3'\Delta^3 y_{-2} + G_4'\Delta^4 y_{-2} + G_5'\Delta^5 y_{-3} + \ldots$$

where $G_1', G_2', \ldots$ are to be determined.

Now,

$$\begin{aligned} y_p &= E^p y_0 \\ &= (1+\Delta)^p y_0 \\ &= 1 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \\ \therefore y_p &= 1 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \end{aligned}$$

Also,

$$\begin{aligned}\Delta y_{-1} &= \Delta E^{-1} y_0 \\ &= \Delta(1+\Delta)^{-1} y_0 \\ &= \Delta\left[1-\Delta+\Delta^2-\Delta^3+\ldots\right] y_0 \\ &= \Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \Delta^4 y_0 + \ldots\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta^2 y_{-1} &= \Delta(\Delta y_{-1}) \\ &= \Delta\left[\Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \Delta^4 y_0 + \ldots\right] \\ &= \Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \ldots\end{aligned}$$

Also,

$$\begin{aligned}\Delta^3 y_{-2} &= \Delta^3 E^{-2} y_0 \\ &= \Delta^3(1+\Delta)^{-2} y_0 \\ &= \Delta^3\left[1-2\Delta+3\Delta^2-4\Delta^3+\ldots\right] y_0 \\ &= \Delta^3 y_0 - 2\Delta^4 y_0 + 3\Delta^5 y_0 - 4\Delta^6 y_0 + \ldots\end{aligned}$$

$$\begin{aligned}\Delta^4 y_{-2} &= \Delta(\Delta^3 y_{-2}) \\ &= \Delta(\Delta^3 y_0 - 2\Delta^4 y_0 + 3\Delta^5 y_0 - 4\Delta^6 y_0 + \ldots) \\ &= \Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \ldots\end{aligned}$$

Substituting on the L.H.S. and R.H.S. of (??) we get

$$\begin{aligned}1 + p\Delta y_0 + & \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \\ & + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots \\ = y_0 & + G_1'\left[\Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \Delta^4 y_0 + \ldots\right] \\ & + G_2'\left[\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \ldots\right] \\ & + G_3'\left[\Delta^3 y_0 - 2\Delta^4 y_0 + 3\Delta^5 y_0 - 4\Delta^6 y_0 + \ldots\right] \\ & + G_4'\left[\Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \ldots\right] \\ & + \ldots\end{aligned}$$

Equating the coefficients of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$ we get,

$$\begin{aligned}G_1 &= p, \\ G_2 &= \frac{p(p+1)}{2!}, \\ G_3 &= \frac{(p+1)p(p-1)}{3!}, \\ G_4 &= \frac{(p+2)(p+1)p(p-1)}{4!}, \\ G_5 &= \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}, \\ &\vdots\end{aligned}$$

Substituting in (??) we obtain the Gauss's Forward Interpolation formula given below

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-3} + \dots$$

17. For the following data the difference table, assuming x_0 at the center, is given in which the terms in orange color are used in Gauss's Backward Interpolation formula.

X	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_9 x_5
Y	y_{-5}	y_{-4}	y_{-3}	y_{-2}	y_{-1}	y_0	y_1	y_2	y_3	y_9 x_5

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$	$\Delta^5 Y$	$\Delta^6 Y$	$\Delta^7 Y$	$\Delta^8 Y$	$\Delta^9 Y$	$\Delta^{10} Y$
x_{-5}	y_{-5}										
		Δy_{-5}									
x_{-4}	y_{-4}		$\Delta^2 y_{-5}$								
		Δy_{-4}		$\Delta^3 y_{-5}$							
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$		$\Delta^4 y_{-5}$						
		Δy_{-3}		$\Delta^3 y_{-4}$		$\Delta^5 y_{-5}$					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$		$\Delta^6 y_{-5}$				
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$		$\Delta^7 y_{-5}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		$\Delta^6 y_{-4}$		$\Delta^8 y_{-5}$		
		Δy_{-1}		**$\Delta^3 y_{-2}$**		**$\Delta^5 y_{-3}$**		**$\Delta^7 y_{-4}$**		**$\Delta^9 y_{-5}$**	
x_0	**y_0**		**$\Delta^2 y_{-1}$**		**$\Delta^4 y_{-2}$**		**$\Delta^6 y_{-3}$**		**$\Delta^8 y_{-4}$**		**$\Delta^{10} y_{-5}$**
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$		$\Delta^7 y_{-3}$		$\Delta^9 y_{-4}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$		$\Delta^8 y_{-3}$		
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$		$\Delta^7 y_{-2}$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		$\Delta^6 y_{-1}$				
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$					
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$						
		Δy_3		$\Delta^3 y_2$							
x_4	y_4		$\Delta^2 y_3$								
		Δy_4									
x_5	y_5										

18. Derive Stirling's interpolation formula for equally spaced arguments.

Answer:

Suppose $\dots, (x_{-3}, y_{-3}), (x_{-2}, y_{-2}), (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$ are some tabulated values of x and y such that (x_0, y_0) is the central data.

For a given x near x_0 suppose,

$$x = x_0 + ph$$

and correponding value of y is y_p

Then we have Gauss's forward and Gauss's backward interpolation formula as given below,

$$\begin{aligned}
y_p = y_0 + p\Delta y_0 + &\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} \\
&+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} \\
&+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2} \\
&+ \dots
\end{aligned}$$

and

$$\begin{aligned}
y_p = y_0 + p\Delta y_{-1} + &\frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} \\
&+ \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} \\
&+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-3} \\
&+ \dots
\end{aligned}$$

Adding and dividing by 2, we get,

$$\begin{aligned}
y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) &+ \frac{p}{2!}\left(\frac{(p+1)+(p-1)}{2}\right)\Delta^2 y_{-1} \\
&+ \frac{(p+1)p(p-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) \\
&+ \frac{(p+1)p(p-1)}{4!}\left(\frac{(p+2)+(p-2)}{2}\right)\Delta^4 y_{-2} \\
&+ \dots
\end{aligned}$$

Simplifying we get the Stirling's Interpolation formula

$$\begin{aligned}
y_p = y_0 + p\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) \\
+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \dots
\end{aligned}$$

Stirling's formula is most desirable when $-\frac{1}{4} \leqslant p \leqslant \frac{1}{4}$

19. **For the following data the difference table assuming x_0 at the center is given in which the terms in blue color are used in Stirling's Interpolation formula.**

X	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_9 x_5
Y	y_{-5}	y_{-4}	y_{-3}	y_{-2}	y_{-1}	y_0	y_1	y_2	y_3	y_9 x_5

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$	$\Delta^5 Y$	$\Delta^6 Y$	$\Delta^7 Y$	$\Delta^8 Y$	$\Delta^9 Y$	$\Delta^{10} Y$
x_{-5}	y_{-5}										
		Δy_{-5}									
x_{-4}	y_{-4}		$\Delta^2 y_{-5}$								
		Δy_{-4}		$\Delta^3 y_{-5}$							
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$		$\Delta^4 y_{-5}$						
		Δy_{-3}		$\Delta^3 y_{-4}$		$\Delta^5 y_{-5}$					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$		$\Delta^6 y_{-5}$				
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$		$\Delta^7 y_{-5}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		$\Delta^6 y_{-4}$		$\Delta^8 y_{-5}$		
		$\mathbf{\Delta y_{-1}}$		$\mathbf{\Delta^3 y_{-2}}$		$\mathbf{\Delta^5 y_{-3}}$		$\mathbf{\Delta^7 y_{-4}}$		$\mathbf{\Delta^9 y_{-5}}$	
x_0	$\mathbf{y_0}$		$\mathbf{\Delta^2 y_{-1}}$		$\mathbf{\Delta^4 y_{-2}}$		$\mathbf{\Delta^6 y_{-3}}$		$\mathbf{\Delta^8 y_{-4}}$		$\mathbf{\Delta^{10} y_{-5}}$
		$\mathbf{\Delta y_0}$		$\mathbf{\Delta^3 y_{-1}}$		$\mathbf{\Delta^5 y_{-2}}$		$\mathbf{\Delta^7 y_{-3}}$		$\mathbf{\Delta^9 y_{-4}}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$		$\Delta^8 y_{-3}$		
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$		$\Delta^7 y_{-2}$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		$\Delta^6 y_{-1}$				
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$					
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$						
		Δy_3		$\Delta^3 y_2$							
x_4	y_4		$\Delta^2 y_3$								
		Δy_4									
x_5	y_5										

20. **The following table gives the values of e^x for certain equidistant values of x. Find the vales of e^x when $x = 0.633$**

x	0.61	0.62	0.63	0.64	0.65	0.66	0.67
$y = e^x$	1.840431	1.858928	1.877610	1.896481	1.915547	1.934792	1.954237

Answer:
For the given data

X	0.61	0.62	0.63	0.64	0.65	0.66	0.67
Y	1.840431	1.858928	1.877610	1.896481	1.915547	1.934792	1.954237

We have to find $y(0.633)$

Following is the diffrence table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$	$\Delta^5 Y$	$\Delta^6 Y$
0.61	1.840431						
		0.018497					
0.62	1.858928		0.000185				
		0.018682		0.000004			
0.63	1.877610		0.000189		0.000002		
		0.018871		0.000006		-0.000024	
0.64	1.896481		0.000195		-0.000022		0.000083
		0.019066		-0.000016		0.000059	
0.65	1.915547		0.000179		0.000037		
		0.019245		0.000021			
0.66	1.934792		0.000200				
		0.019445					
0.67	1.954237						

Here, we have h = 0.62 - 0.61 = 0.01

We shall use Gauss' Forward Interpolation formula

Here, 0.633 lies in [0.63, 0.64]

So, we can appropriately choose 0.63 as central argument

Take $x_0 = 0.63$
Let $x = x_0 + ph$

Therefore $p = \dfrac{0.633 - 0.63}{0.01} = 0.3000$

Substituting for p and differences in the **Gauss' Forward Interpolation formula**

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2} + \ldots$$

we get $y(0.633) = y_p = 1.883251$

> 21. **Using Gauss's forward interpolation formula find $f(32)$, given that** $f(25) = 0.2707,\ f(30) = 0.3027,\ f(35) = 0.3386,\ f(40) = 0.3794$

Answer:
For the given data

X	25	30	35	40
Y	0.2707	0.3027	0.3386	0.3794

We have to find $y(32)$

Following is the diffrence table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$
25	0.2707			
		0.0320		
30	0.3027		0.0039	
		0.0359		0.0010
35	0.3386		0.0049	
		0.0408		
40	0.3794			

Here, we have h = 30 - 25 = 5

We shall use Gauss' Forward Interpolation formula

Here, 32 lies in [30, 35]

So, we can appropriately choose 30 as central argument

Take $x_0 = 30$
Let $x = x_0 + ph$

Therefore $p = \dfrac{32 - 30}{5} = 0.4$

Substituting for p and differences in the **Gauss' Forward Interpolation formula**

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2} + \dots$$

we get $y(32) = y_p = 0.316536$

22. **By using Gauss's backward interpolation formula find a cubic polynomial $f(x)$ given that**

$$f(1) = -1,\ f(2) = 11,\ f(3) = 35,\ f(4) = 77,\ \text{and}\ , f(5) = 143$$

Hence find $f(0)$ and $f(6)$

Answer:
Given data can be tabulated as follows

X	1	2	3	4	5
Y	-1	11	35	77	143

Following is the diffrence table of the data

X	Y	ΔY	Δ²Y	Δ³Y	Δ⁴Y
1	-1				
		12			
2	11		12		
		24		6	
3	35		18		0
		42		6	
4	77		24		
		66			
5	143				

Here, all the arguments are equally spaced and their common difference is
$h = 2 - 1 = 1$

For applying Gauss's forward interpolation formula we choose cetral observation $x_0 = 3$
Let $x = x_0 + ph = 3 + p(1) = p + 3$

Substituting $x - 3$ for p and differences of y in the Gauss's forward formula we get

$$\begin{aligned}
f(x) &= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} \\
&\quad + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} \\
&\quad + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-3} + \dots \\
&= 35 + (x-3)(24) + \frac{(x-2)(x-3)}{2!}(18) + \frac{(x-2)(x-3)(x-4)}{3!}(6) \\
&= 35 + 24x - 72 + (9x^2 - 45x + 54) + (x^3 - 9x^2 + 26x - 24) \\
&= x^3 + 5x - 7
\end{aligned}$$

Thus, the required polynomial is $y(x) = x^3 - 5x - 7$

Therefore, $f(0) = -7$ and
$f(6) = 6^3 - 5(6) - 7 = 216 - 30 - 7 = 179$

23. By using Gauss's backward interpolation formula find $\sqrt{12525}$
$\sqrt{12500} = 111.8034, \sqrt{12} = 111.8481, \sqrt{12} = 111.8928, \sqrt{12} = 111.9375, \sqrt{12} = 111.9822$

Answer:
For the given data

X	12500	12510	12520	12530	12540
Y	111.8034	111.8481	111.8928	111.9375	111.9822

We have to find $y(12525)$

Following is the diffrence table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$
12500	111.8034				
		0.0447			
12510	111.8481		-0.0000		
		0.0447		0.0000	
12520	111.8928		0.0000		-0.0000
		0.0447		-0.0000	
12530	111.9375		0.0000		
		0.0447			
12540	111.9822				

Here, we have h = 12510 - 12500 = 10

We shall use Gauss' Backward Interpolation formula

Here, 12525 lies in [12520, 12530]

So, we can appropriately choose 12530 as central argument

Take $x_0 = 12530$
Let $x = x_0 + ph$

Therefore $p = \dfrac{12525 - 12530}{10} = -0.5$

Substituting for p and differences in the **Gauss' Backward Interpolation formula**

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-3} + \ldots$$

we get $y(12525) = y_p = 111.91515 \approx 111.9152$

24. Use Stirling's formula to find u_{32}, given that
$u_{20} = 14.035,\ u_{25} = 13.674,\ u_{30} = 13.257,\ u_{35} = 12.734,\ u_{40} = 12.089$ **and** $u_{45} = 11.309$

Answer:

X	20	25	30	35	40	45
Y	14.035	13.674	13.257	12.734	12.089	11.309

We have to find $y(32)$

Following is the diffrence table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$	$\Delta^5 Y$
20	14.035					
		-0.361				
25	13.674		-0.056			
		-0.417		-0.050		
30	13.257		-0.106		0.034	
		-0.523		-0.016		-0.031
35	12.734		-0.122		0.003	
		-0.645		-0.013		
40	12.089		-0.135			
		-0.780				
45	11.309					

Here, we have h = 25 - 20 = 5

We shall use Stirling's Interpolation formula

Here, 32 lies in [30, 35]

So, we can appropriately choose 30 as central argument

Take $x_0 = 30$
Let $x = x_0 + ph$

Therefore $p = \dfrac{32 - 30}{5} = 0.40$

Substituting for p and differences in the **Striling' Interpolation formula**

$$y_p = y_0 + p\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{p^2}{2}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots$$

we get $y(32) = y_p = 13.06201094$

25. **Find the cubic polynomial which takes the following values**

$$y(0) = 1,\ y(1) = 0,\ y(2) = 1,\ y(3) = 10$$

Hence find the value of $y(4) - y(0.5)$

Answer:
Given data can be tabualted as follows

x	0	1	2	3
y	0	0	1	10

Following is the difference table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

Here, all the arguments are equally spaced and their common difference is
$h = 1 - 0 = 1$

Let $x = x_0 + ph = 0 + p(1) = p$

Substituting x for p and differences of y in the Newton's forward difference formula we get

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \frac{p(p-1)\ldots(p-n+1)}{n!}\Delta^n y_0$$

$$= 1 + x(-1) + \frac{x(x-1)}{2!}(2) + \frac{x(x-1)(x-2)}{3!}(6)$$

$$= 1 - x + x(x-1) + x(x-1)(x-2)$$
$$= 1 - x + x^2 - x + x(x^2 - 3x + 2)$$
$$= 1 - x + x^2 - x + x^3 - 3x^2 + 2x$$

$$= x^3 - 2x^2 + 1$$

Thus, the required polynomial is $y(x) = x^3 - 2x^2 + 1$

Now, $y(4) - y(0.5) = (4^3 - 2(4^2) + 1) - ((0.5)^3 - 2(0.5^2) + 1) = 33 - 0.625 = 32.375$

26. **Bessel's Interpolation Formula**

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{p(p-1)(p-\frac{1}{2})}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \ldots$$

NOTE : Bessel's formula is most desirable when $\frac{1}{4} \leqslant p \leqslant \frac{3}{4}$

27. **Everett's Interpolation Formula**

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + ...$$
$$+ py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + ...$$

where $q = 1 - p$

28. **Let** $y = g(x)$ **be a function such that**

$$g(20) = 2854,\ g(24) = 3162,\ g(28) = 3544,\ g(32) = 3992$$

Use Bessel's formula to obtain $g(25)$.

Answer:

For the given data

X	20	24	28	32
Y	2854	3162	3544	3992

We have to find $y(25)$

Following is the diffrence table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$
20	2854			
		308		
24	3162		74	
		382		-8
28	3544		66	
		448		
32	3992			

Here, we have h = 24 - 20 = 4

We shall use Bessel's Interpolation formula

Here, 25 lies in [24, 28]

So, we can appropriately choose 24 as central argument

Take $x_0 = 24$

Let $x = x_0 + ph$

Therefore $p = \dfrac{25-24}{4} = 0.25$

Substituting for p and differences in the **Bessel's Interpolation formula**

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{p(p-1)(p-\frac{1}{2})}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \ldots$$

we get $y(25) = y_p = 3250.875 \approx 3251$

29. **Let $y = g(x)$ be a function such that**

$$g(20) = 2854,\ g(24) = 3162,\ g(28) = 3544,\ g(32) = 3992$$

Use Everett's formula to obtain $g(25)$.

Answer:

For the given data

X	20	24	28	32
Y	2854	3162	3544	3992

We have to find $y(25)$

Following is the diffrence table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$
20	2854			
		308		
24	3162		74	
		382		-8
28	3544		66	
		448		
32	3992			

Here, we have h = 24 - 20 = 4

We shall use Everett's Interpolation formula

Here, 25 lies in [24, 28]

So, we can appropriately choose 24 as central argument

Take $x_0 = 24$

Let $x = x_0 + ph$

Therefore $p = \dfrac{25 - 24}{4} = 0.25$

Substituting for p, q and differences in the **Everett's Interpolation formula**

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + ...$$
$$+ py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + ...$$

where $q = 1 - p$

we get $y(25) = y_p = 3250.875 \approx 3251$