F.Y.B.Sc.-Sem 2

US02CMTH21 (T)

Algebra

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US02CMTH21 (T)- UNIT : IV

• Solution of linear equations

1. Define Solution of General Linear System of equations.

A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ has the general form

$a_{11}x_1 + a_{12}x_2, \ldots, a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2, \ldots, a_{2n}x_n = b_2$
$\cdots$
$a_{m1}x_1 + a_{m2}x_2, \ldots, a_{mn}x_n = b_n$

If $b_1 = b_2 = \cdots = b_n = 0$ then the system of linear equations is called a Homogeneous system of Linear equations. In case atleast one of $b_i, i = 1, 1 \ldots n$ is non zero then the system is called a Non-Homogeneous system of Linear equations.

2. Using Gauss Elimination method solve the following system of equations, if possible.

[1] $2x + y + z = 0,\ 3x + 2y + 3z = 18,\ x + 4y + 9z = 16$;

$$[A0|I] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right] R_{13}(-1),$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & -16 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right] R_{21}(-3), R_{31}(-1),$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & -16 \\ 0 & 11 & 27 & 66 \\ 0 & 7 & 17 & 32 \end{array}\right] R_2(1/11),$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & -16 \\ 0 & 1 & \frac{27}{11} & 6 \\ 0 & 7 & 17 & 32 \end{array}\right] R_{32}(-7),$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & -16 \\ 0 & 1 & \frac{27}{11} & 6 \\ 0 & 0 & -\frac{2}{11} & -10 \end{array}\right] R_{3}(-11/2),$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & -16 \\ 0 & 1 & \frac{27}{11} & 6 \\ 0 & 0 & 1 & 55 \end{array}\right]$$

Here the Left Matrix is converted to its Row Echelon form.
So, the reduced system of eauation is
$x - 3y - 8z = -16$ - - - (i)
$y + \frac{27}{11}z = 6$ - - - (ii)
$z = 6$ - - - (iii)
Substituting $z = 55$ in (ii) we get $y = -129$
Substituting $y = -129$ in $z = 55$ (i)
$x = 37$
Therefore the solution is

$$x = 37,\ y = -129,\ z = 55$$

[2] $x - 2y + w = 3,\ -x + 2y + z - \frac{1}{2}w = -7,\ 4x - 8y + 6z + 7w = -3$

$$[A1|I] = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 3 \\ -1 & 2 & 1 & -\frac{1}{2} & -7 \\ 4 & -8 & 6 & 7 & -3 \end{array}\right] R_{21}(1), R_{31}(-4),$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & \frac{1}{2} & -4 \\ 0 & 0 & 6 & 3 & -15 \end{array}\right] R_{32}(-6),$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & \frac{1}{2} & -4 \\ 0 & 0 & 0 & 0 & 9 \end{array}\right]$$

The system has no solution, as the last row on the left is a ZERO row but the last element on the Right is non-zero. Hence the system is inconsistent.

$$[\mathbf{3}]\ \ -\frac{1}{x}+\frac{3}{y}+\frac{4}{z}=30,\ \ \frac{3}{x}+\frac{2}{y}-\frac{1}{z}=9,\ \ \frac{2}{x}-\frac{1}{y}+\frac{2}{z}=10$$

$$[A3|I]=\left[\begin{array}{rrr|r} -1 & 3 & 4 & 30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right] R_1(-1),$$

$$\sim\left[\begin{array}{rrr|r} 1 & -3 & -4 & -30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right] R_{21}(-3), R_{31}(-2),$$

$$\sim\left[\begin{array}{rrr|r} 1 & -3 & -4 & -30 \\ 0 & 11 & 11 & 99 \\ 0 & 5 & 10 & 70 \end{array}\right] R_2(1/11),$$

$$\sim\left[\begin{array}{rrr|r} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 5 & 10 & 70 \end{array}\right] R_{32}(-5),$$

$$\sim\left[\begin{array}{rrr|r} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 5 & 25 \end{array}\right] R_3(1/5),$$

$$\sim\left[\begin{array}{rrr|r} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$x=\frac{1}{2},\ y=\frac{1}{4},\ z=\frac{1}{5}.$

$$[\mathbf{4}]\ \ 2x+2y+2z=0,\ \ -2x+5y+2z=1,\ \ 8x+y+4z=-1$$

$$[A|I]=\left[\begin{array}{rrr|r} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right] R_1(1/2),$$

$$\sim\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right] R_{21}(2), R_{31}(-8),$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right] R_2(1/7),$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & -7 & -4 & -1 \end{array}\right] R_{32}(7),$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

As the last row elmenents of Left and Right row both are zero the system has infinitely many solution

[5] $4x + 3y - z = 0,\ 3x + 4y + z = 0,\ 5x + y - 4z = 0$

$$[A4|I] = \left[\begin{array}{ccc|c} 4 & 3 & -1 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & -4 & 0 \end{array}\right] R_{12}(-1),$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & -4 & 0 \end{array}\right] R_{21}(-3), R_{31}(-5),$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 6 & 6 & 0 \end{array}\right] R_2(1/7),$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 6 & 6 & 0 \end{array}\right] R_{32}(-6),$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3. **Consider the system of equations** $x+y+z = 6,\ x+2y+3z = 10,\ x+2y+\lambda z = \mu$**. For what values of** λ **and** μ **does the system have (i) no solution (ii) unique solution (iii) infinite solutions?**

$$[A|I] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] R_{21}(-1), R_{31}(-1),$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right] R_{32}(-1),$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$$

4. **What condition must b_1, b_2 and b_3 satisfy in order for the system of equations $x_1 + 2x_2 + 3x_3 = b_1,\ 2x_1 + 5x_2 + 3x_3 = b_2,\ x_1 + 8x_3 = b_3$ to be consistent?**

$$[A|I] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{array}\right] R_{21}(-2), R_{31}(-1),$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & -3 & b_2 - 2b_1 \\ 0 & -2 & 5 & b_3 - b_1 \end{array}\right] R_{32}(2),$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & -3 & b_2 - 2b_1 \\ 0 & 0 & -1 & b_3 - 2b_2 - 5b_1 \end{array}\right] R_3(-1),$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & -3 & b_2 - 2b_1 \\ 0 & 0 & 1 & 5b_1 + 2b_2 - b_3 \end{array}\right]$$

5. **Find the value λ so that the following equations have a non-trivial solution $2x + y + 2z = 0, x + y + 3z = 0, 4x + 3y + \lambda z = 0$.**

$$[A|I] = \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \end{array}\right] R_{12}(-1),$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \end{array}\right] R_{21}(-1), R_{31}(-4),$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 3 & \lambda+4 & 0 \end{array}\right] R_{32}(-3),$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & \lambda-8 & 0 \end{array}\right]$$

[1] Cayley-Hamilton Theorem.

Define the following terms.

[1] Singular Matrix

A square matrix A is said to be singular if $|A| = 0$

[2] Non-singular Matrix

A square matrix A is said to be Non-singular if $|A| \neq 0$

[3] Characteristic Matrix

For a square matrix A the matrix $A - xI$ is called its characteristic matrix.

[4] Characteristic Equation of a Matrix

For a square matrix A, an equation given by $|A - xI| = 0$ is called its characteristic Equation.

7. State and prove *Cayley-Hamilton theorem*

Suppose, A is a square matrix and let

$$|A - xI| = a_0 + a_1x + a_2x^2 + ... + a_nx^n = 0$$

be the characteristic equation of A
Now, suppose,

$$adj.(A - xI) = B_0 + B_1x + B_2x^2 + ... + B_{n-1}x^{n-1}$$

As, $(A - xI).adj(A - xI) = |A - xI|.I$, we get

$$(A - xI)(B_0 + B_1x + B_2x^2 + ... + B_{n-1}x^{n-1}) = (a_0 + a_1x + a_2x^2 + ... + a_nx^n)I$$
$$\therefore \quad (A - xI)(B_0 + B_1x + B_2x^2 + ... + B_{n-1}x^{n-1}) = a_0I + a_1Ix + a_2Ix^2 + ... + a_nIx^n$$

Comparing respective coefficients of powers of x on both the sides, we get

$$\begin{aligned} AB_0 &= a_0I \\ AB_1 - B_0 &= a_1I \\ AB_2 - B_1 &= a_2I \\ &... \\ &... \\ -B_{n-1} &= a_nI \end{aligned}$$

Pre-multiplying theses successively with $I, A, A^2, A^3, ..., A^n$ and adding them we get

$$a_0I + a_1A + a_2A^2 + ... + a_nA^n = O$$

This proves *Cayley-Hamilton Theorem*

8. **Find the characteristic equation of the matrix** $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ **and verify that it is satisfied by** A **and hence obtain** A^{-1}**.**

Finding Characteristic equation:
The characteristic equation is given by $|A - xI| = 0$.
Now,

$$\begin{aligned} |A - xI| = 0 &\Longrightarrow \begin{vmatrix} 2-x & -1 & 1 \\ -1 & 2-x & -1 \\ 1 & -1 & 2-x \end{vmatrix} = 0 \\ &\Longrightarrow (2-x)[(2-x)^2 - 1] + 1[-(2-x) + 1] + 1[1 - (2-x)] = 0 \\ &\Longrightarrow (2-x)^3 - 3(2-x) + 2 = 0 \\ &\Longrightarrow (8 - 12x + 6x^2 - x^3) - 6 + 3x + 2 = 0 \\ &\Longrightarrow -x^3 + 6x^2 - 9x + 4 = 0 \end{aligned}$$

Thus, the characteristics equation is $x^3 - 6x^2 + 9x - 4 = 0$

Now we show that the characteristic equation is satisfied by A:

$A^3 - 6A^2 + 9A - 4I =$

$$\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & 6 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = O$$

A^{-1} can be calculated as follows

$$A^{-1} = \frac{1}{4}A^2 - \frac{8}{2}A + \frac{9}{4}I = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

9. **Show that the matrix** $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **satisfies** *Cayley-Hamilton* **theorem. Hence or otherwise obtain** A^{-1} **and** A^{-2}**.**

Characteristic equation $\lambda^3 + \lambda^2 - 5\lambda - 5 = 0$

The characteristic equation is satisfied :

$A^3 + A^2 - 5A - 5I =$

$$\begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = O$$

Also A^{-1} and A^{-2} can be calculated given as follows

$$A^{-1} = \frac{1}{5}(A^2 + A - 5I) = \frac{1}{5} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and}$$

10. **Verify** *Cayley-Hamilton* **theorem for the matrix** $\begin{bmatrix} 0 & 1 & 2 \\ 3 & -3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$**. Hence find its inverse if possible**

Characteristic equation $\lambda^3 + 4\lambda^2 - 4\lambda - 17 = 0$

The characteristic equation is satisfied :

$A^3 + 4A^2 - 4A - 17I =$

$$\begin{bmatrix} -3 & 8 & 8 \\ 40 & -51 & 16 \\ -4 & 16 & -7 \end{bmatrix} + 4 \begin{bmatrix} 5 & -1 & 0 \\ -7 & 14 & -2 \\ 2 & -3 & -5 \end{bmatrix} - 4 \begin{bmatrix} 0 & 1 & 2 \\ 3 & -3 & 2 \\ 1 & 1 & -1 \end{bmatrix} - 17 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = O$$

Also A^{-1} is given by

$$A^{-1} = \frac{1}{17}(A^2 + 4A - 4I) = \frac{1}{17}\begin{bmatrix} 1 & 3 & 8 \\ 5 & -2 & 6 \\ 6 & 1 & -3 \end{bmatrix}$$

11. Verify *Cayley-Hamilton* theorem for the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$. Hence find its inverse if possible

Characteristic equation $\lambda^3 - \lambda^2 - 18\lambda - 30 = 0$

The characteristic equation is satisfied :

$A^3 - A^2 - 18A - 30I =$

$$\begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix} - \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} - 18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} - 30\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = O$$

Also A^{-1} is given by

$$A^{-1} = \frac{1}{30}(A^2 - A - 18I) = \frac{1}{30}\begin{bmatrix} -1 & 1 & 11 \\ 1 & -4 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

12. Verify *Cayley-Hamilton* theorem for $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and use it to find the simplified form of $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I_3$.

For the given matrix A the characterstic equation is given by,

$$|A - xI| = 0$$

$$\begin{vmatrix} -x+2 & 1 & 1 \\ 0 & -x+1 & 0 \\ 1 & 1 & -x+2 \end{vmatrix} = 0$$

$$x^3 - 5\,x^2 + 7\,x - 3 = 0$$

Now, we verify whether matrix A satisfies its characteristic equation

$$A^3 - 5A^2 - 7A - 3I = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - \begin{bmatrix} 25 & 20 & 20 \\ 0 & 5 & 0 \\ 20 & 20 & 25 \end{bmatrix} - \begin{bmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since,

$$A^3 - 5A^2 - 7A - 3I = O$$

matrix A satisfies its characterstic equation.
Thus, Cayley-Hamilton theorem is verified for the matrix A

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[1] Properties of Eigen Values and Eigen Vectors.

Define the following terms.

[1] Characteristic Vector of a matrix

Any non-zero vector X is said to be a characteristic vector or Eigen vector of a square matrix A if there exists a real number λ such that $AX = \lambda X$. Here λ is known as a characteristic root or Eigen root of the matrix correcponding to the characteristic vector X

[2] Characteristic Root of a Matrix

Any non-zero vector X is said to be a characteristic vector or Eigen vector of a square matrix A if there exists a real number λ such that $AX = \lambda X$. Here λ is known as a characteristic root or Eigen root of the matrix correcponding to the characteristic vector X

[3] Orthogonal Matrix

For a square matrix A if $AA' = I$ then A is called an orthogonal matrix.

14.

15. **For square matrices of same order A and X the product $X^\theta AX$ is called Hermitan Form.**

For a square matrix A if $AA^\theta = A^\theta A = I$ then A is called a Unitary matrix.

16. **A Hermitan Form always assumes a real value.**

For a square matrix A if $AA^\theta = A^\theta A = I$ then A is called a Unitary matrix.

17.

18. **Prove that the characteristic roots of a real symmetric matrix are all real.**

Let A be a real symmetric matrix.
Therefore, we have $A' = A$ and $\overline{A} = A$
Now,

$$\begin{aligned} A^\theta &= \left(\overline{A}\right)' \\ &= A' \\ &= A \end{aligned}$$

Therefore A is a Hermitian matrix also.
By a theorem, the characteristic roots of every Hermitian matrix are all real.
Hence, the characteristic roots of every real symmetric matrix are all real.

19. **Prove that characteristic roots of a Skew-Hermitian matrix are either zero or pure imaginary numbers.**

Let λ be a characteristic root of a Skew-Hermitian matrix A and $X \neq O$ be corresponding characteristic vector.
As A is a Skew-Hermitian matrix we have $A^\theta = -A$
Therefore,

$$AX = \lambda X$$

Multiplying with i we get,

$$iAX = i\lambda X$$

$$(iA)X = (i\lambda)X$$

Therefore, $i\lambda$ is a characteristic root of iA.
Now,

$$\begin{aligned} (iA)^\theta &= \bar{i}A^\theta \\ &= -i(-A) \\ &= iA \end{aligned}$$

Therefore iA is a Hermitian matrix.
By a theorem, the characteristic roots of every Hermitian matrix are all real.
Therefore, all the characteristic roots $i\lambda$ of iA are real.

Hence, for every characteristic root $i\lambda$ to be a real number, either $\lambda = 0$ or λ is purely imaginary.

20. Prove that characteristic roots of a real Skew-Symmetric matrix are either zero or pure imaginary numbers.

Let A be a real Skew-Symmetric matrix.
Therefore, we have $A' = -A$ and $\overline{A} = A$
Now,

$$\begin{aligned} A^{\theta} &= \left(\overline{A}\right)' \\ &= A' \\ &= -A \end{aligned}$$

Therefore A is a Skew-Hermitian matrix also.
By a theorem, the characteristic roots of every Skew-Hermitian matrix are either zero or pure imaginary.
Hence, the characteristic roots of a real Skew-Symmetric matrix are either zero or pure imaginary numbers.

21. Prove that the modulus of a characteristic root of a unitary matrix is unity.

Let A be a Unitary matrix.

$$A^{\theta}A = AA^{\theta} = I$$

Now, if λ is a characteristic root of A and $X \neq O$ is corresponding characteristic vector then,

$$AX = \lambda X \quad \text{- - - - (1)}$$

$$\begin{aligned} \therefore (AX)^{\theta} &= (\lambda X)^{\theta} \\ \therefore X^{\theta}A^{\theta} &= \overline{\lambda}X^{\theta} \\ \therefore \left(X^{\theta}A^{\theta}\right)(AX) &= \left(\overline{\lambda}X^{\theta}\right)(\lambda X) \quad \text{(using (1))} \\ \therefore X^{\theta}\left(A^{\theta}A\right)X &= \left(\lambda\overline{\lambda}\right)X^{\theta}X \\ \therefore X^{\theta}IX &= \left(\lambda\overline{\lambda}\right)X^{\theta}X \\ \therefore X^{\theta}X &= \left(\lambda\overline{\lambda}\right)X^{\theta}X \\ \therefore \left(1-\lambda\overline{\lambda}\right)X^{\theta}X &= 0 \end{aligned}$$

As $X \neq O$, we have $X^{\theta}X \neq 0$.
Therefore, we have,

$$1 - \lambda\overline{\lambda} = 0$$

$$\therefore \lambda\overline{\lambda} = 1$$

$$\therefore |\lambda|^2 = 1$$

$$\therefore |\lambda| = 1$$

Thus, modulus of a characteristic root of a unitary matrix is unity.

22. Prove that the modulus of a each characteristic root of an orthogonal matrix is unity.

Let A be an orthogonal matrix.
Therefore, we have $AA' = A'A = I$.
Also, as A is real, $\overline{A} = A$
Now, $A^\theta = \left(\overline{A}\right)' = A'$
Therefore,

$$AA^\theta = AA' = I$$

Hence A is a Unitary matrix also.
By a theorem, the modulus of a characteristic root of a Unitary matrix is unity.
Hence, the modulus of a each characteristic root of an orthogonal matrix is also unity.

23. If S is a real skew-symmetric matrix then prove that $I - S$ is non-singular and the matrix $A = (I + S)(I - S)^{-1}$ is orthogonal

Here, S is a real skew-symmetric matrix.
Now, if $I - S$ is a singular matrix then $|S - I| = 0$

But then 1 is a characteristic root of S which is not possible as S being real skew-symmetric it can have only zeros or purely imaginary roots.
Thus, $I - S$ is non-singular

Next, we show that $A = (I + S)(I - S)^{-1}$ is orthogonal

Now, $A' = [(I - S)^{-1}]'(I + S)' = [(I - S)']^{-1}(I + S)'$

But, $(I - S)' = I' - S' = I + S$ and $(I + S)' = I' + S' = I - S$
$\therefore \quad A' = (I + S)^{-1}(I - S)$

Now,

$$A'A = (I + S)^{-1}(I - S)(I + S)(I - S)^{-1}$$

$$= (I + S)^{-1}(I + S)(I - S)(I - S)^{-1}$$

$$\therefore \quad A'A = I$$

Therefore A is an orthogonal matrix

24. **Prove that every orthogonal matrix A can be expressed as $A = (I+S)(I-S)^{-1}$ by a suitable choice of real skew-symmetric matrix S provided that -1 is not a characteristic root of A**

To prove the theorem it is sufficient to show that for an orthogonal matrix A such that -1 is not a characeristic root of A, such that $A = (I + S)(I - S)^{-1}$ determines a skew-symmetric matrix S. Now,

$$\begin{aligned} A = (I + S)(I - S)^{-1} &\Rightarrow A(I - S) = I + S \\ &\Rightarrow A - AS = I + S \\ &\Rightarrow A - I = (A + I)S \text{ - - - } (1) \end{aligned}$$

Since -1 is not a characteristic root of A we have $|A - (-1)I| \neq 0$.
Therefore,

$$|A + I| \neq 0$$

Hence, $(A + I)^{-1}$ exists.
Therefore, premultiplying with $(A + I)^{-1}$ on both sides of (2) we get,

$$S = (A + I)^{-1}(A - I)$$

This estabishes existence of S. Finally we show that S is a real skew symmetric matrix.

$$\begin{aligned} S' &= [(A + I)^{-1}(A - I)]' \\ &= (A - I)'[(A + I)^{-1}]' \\ &= (A - I)'[(A + I)']^{-1} \\ &= (A' - I)(A' + I)^{-1} \\ &= (A' + I)^{-1}(A' - I) \\ &= (A' + A'A)^{-1}(A' - A'A) \\ &= [A'(I + A)]^{-1}[A'(I - A)] \\ &= (I + A)^{-1}(A')^{-1}A'(I - A) \\ &= (I + A)^{-1}(I - A) \\ &= -(A + I)^{-1}(A - I) \\ &= -S \end{aligned}$$

Hence, S is a skew-symmetric matrix.

25. **Show that a characteristic vector X, corresponding to a characteristic root λ of a martix A is also a characteristic vector of every matrix $f(A)$; $f(x)$ being any scalar polynomial, and the corresponding root for $f(A)$ is $f(\lambda)$. In general show that if $g(x) = \dfrac{f_1(x)}{f_2(x)}$; where $|f_2(A)| \neq O$ then $g(\lambda)$ is a characteristic root of $g(A) = f_1(A)\{f_2(A)\}^{-1}$.**

Let λ be a characteristic root and $X \neq O$ be corresponding characteristic vector of a matrix A. Therefore,

$$AX = \lambda X$$

Now,

$$A^2X = A(AX) = A(\lambda X) = \lambda(AX) = \lambda^2 X$$

Repeating the process k times, we get,

$$A^kX = \lambda^k X$$

If $f(x) = a_0x + a_1x + a_2x^2 + \cdots + a_mx^m$ is a scalar polynomial then we have,

$$\begin{aligned} f(A)X &= \left(a_0I + a_1A + a_2A^2 + \cdots + a_mA^m\right) X \\ &= a_0X + a_1AX + a_2A^2X + \cdots + a_mA^mX \\ &= a_0X + a_1\lambda X + a_2\lambda^2X + \cdots + a_m\lambda^mX \\ &= \left(a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_m\lambda^m\right) X \\ \therefore f(A)X &= f(\lambda)X \end{aligned}$$

Therefore, $f(\lambda)$ is a characteristic root of $f(A)$ corresponding to characteristic vector X.

Hence, $f_1(\lambda)$ and $f_2(\lambda)$ are characterstic roots of $f_1(A)$ and $f_2(A)$ respectively.
Therefore,

$$f_1(A)X = f_1(\lambda)X \quad \text{and} \quad f_2(A)X = f_2(\lambda)X$$

Now, if $|f_2(A)| \neq 0$ then $f_2(A)$ is a non-singular matrix and hence characteristic roots of $f_2(A)$ are non-zero.
Therefore,

$$f_2(\lambda) \neq 0$$

Hence, we also have $\{f_2(A)\}^{-1} X = \{f_2(\lambda)\}^{-1} X$

Now, if $g(A) = f_1(A)\{f_2(A)\}^{-1}$ then

$$\begin{aligned} g(A)X &= f_1(A)\left\{[f_2(A)]^{-1}X\right\} \\ &= f_1(A)\left\{[f_2(\lambda)]^{-1}X\right\} \\ &= \{f_2(\lambda)\}^{-1}(f_1(A)X) \\ &= \{f_2(\lambda)\}^{-1}(f_1(\lambda)X) \\ &= f_1(\lambda)\{f_2(\lambda)\}^{-1}X \\ &= g(\lambda)X \end{aligned}$$

Thus, $g(\lambda)$ is a characteristic root and X is corresponding characteristic vector of $g(A) = f_1(A)[f_2(A)]^{-1}$

26. **Show that the two matrices A and $P^{-1}AP$ have the same characteristic roots.**

Let A be a square matrix and P be a non-singular matrix of same order.
Suppose $B = P^{-1}AP$.
Now,

$$B - xI = P^{-1}AP - xI = P^{-1}AP - P^{-1}(xI)P = P^{-1}(A - xI)P$$

Therefore,

$$\begin{aligned}|B-xI| &= \left|P^{-1}(A-xI)P\right| \\ &= \left|P^{-1}\right||A-xI||P| \\ &= \left|P^{-1}\right||P||A-xI| \\ &= \left|P^{-1}P\right||A-xI| \\ &= |I||A-xI| \\ &= |A-xI|\end{aligned}$$

Therefore,

$$|B-xI|=0 \iff |A-xI|=0$$

Therefore, $P^{-1}AP$ and A have same characteristic equations. Hence, $P^{-1}AP$ and A have same characteristic roots.

27. **Show that the characteristic roots of A^θ are the conjugates of the characteristic roots of A.**

Let λ be a characteristic root and $X \neq O$ be corresponding characteristic vector of a square matrix A.
Now,

$$\begin{aligned}|A^\theta-\overline{\lambda}I| &= |(A-\lambda I)^\theta| \\ &= |\overline{(A-\lambda I)'}| \\ &= |\overline{(A-\lambda I)}|\end{aligned}$$

Therefore,

$$|A^\theta-\overline{\lambda}I|=0 \iff |\overline{(A-\lambda I)}|=0 \iff |A-\lambda I|=0$$

Hence, $\overline{\lambda}$ is a characteristic root of A^θ whenever λ is a characteristic root of A.

28. **Find the characteristic roots and characteristic vectors of**

[1] $\begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}$

For the given matrix A the characterstic equation is given by,

$$|A-xI|=0$$

$$\begin{vmatrix} -x-4 & 8 & -12 \\ 6 & -x-6 & 12 \\ 6 & -8 & -x+14 \end{vmatrix} = 0$$

$$x^3 - 4\,x^2 + 4\,x = 0$$

$$(x-2)^2 x = 0$$

The eigen values of A are

$$\lambda = 2, 0$$

<u>Finding eigen vectors for the eigen value $\lambda = 2$</u>

Corresponding to $\lambda = 2$ we have the following matrix equation

$$(A - (2)I)X = O$$

$$\begin{bmatrix} -6 & 8 & -12 \\ 6 & -8 & 12 \\ 6 & -8 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results in the following system of linear equations

$$\begin{aligned} -6\,x + 8\,y - 12\,z = 0 \quad & \text{- - - (1)} \\ 6\,x - 8\,y + 12\,z = 0 \quad & \text{- - - (2)} \\ 6\,x - 8\,y + 12\,z = 0 \quad & \text{- - - (3)} \end{aligned}$$

Here equations (1),(2) and (3) are linealy dependent So we consider any one of them. Let us consider

$$-6\,x + 8\,y - 12\,z = 0$$

$$x = \frac{4}{3}\,y - 2\,z$$

Therefore the eigen vectors corresponding to $\lambda = 2$ are given by

$$X = \begin{bmatrix} \frac{4}{3}\,y - 2\,z \\ y \\ z \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{4}{3}\,y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -2\,z \\ 0 \\ z \end{bmatrix}$$

$$X = y \begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

where $y, z \in R - 0$

Finding eigen vectors for the eigen value $\lambda = 0$
Corresponding to $\lambda = 0$ we have the following matrix equation

$$(A-(0)I)X = O$$

$$\begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results in the following system of linear equations

$$\begin{aligned} -4\,x + 8\,y - 12\,z = 0 \quad & \text{- - - (1)} \\ 6\,x - 6\,y + 12\,z = 0 \quad & \text{- - - (2)} \\ 6\,x - 8\,y + 14\,z = 0 \quad & \text{- - - (3)} \end{aligned}$$

No two equations are mutually linearly dependent Solving equations (1) and (2) using Cramers rule

$$\frac{x}{\begin{vmatrix} 8 & -12 \\ -6 & 12 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -4 & -12 \\ 6 & 12 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -4 & 8 \\ 6 & -6 \end{vmatrix}}$$

$$\frac{x}{1} = \frac{-y}{-1} = \frac{z}{-1}$$

Therefore the eigen vectors corresponding to $\lambda = 0$ are given by

$$X = k \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

where $k \in R - 0$

[2] $\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

For the given matrix A the characterstic equation is given by,

$$|A - xI| = 0$$

$$\begin{vmatrix} -x-2 & -8 & -12 \\ 1 & -x+4 & 4 \\ 0 & 0 & -x+1 \end{vmatrix} = 0$$

$$x^3 - 3\,x^2 + 2\,x = 0$$

$$(x-1)(x-2)x = 0$$

The eigen values of A are

$$\lambda = 1, 2, 0$$

Finding eigen vectors for the eigen value $\lambda = 1$
Corresponding to $\lambda = 1$ we have the following matrix equation

$$(A - (1)I)X = O$$

$$\begin{bmatrix} -3 & -8 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results in the following system of linear equations

$$-3\,x - 8\,y - 12\,z = 0 \quad \text{- - - (1)}$$
$$x + 3\,y + 4\,z = 0 \quad \text{- - - (2)}$$

No two equations are mutually linearly dependent Solving equations (1) and (2) using Cramers rule

$$\frac{x}{\begin{vmatrix} -8 & -12 \\ 3 & 4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -3 & -12 \\ 1 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & -8 \\ 1 & 3 \end{vmatrix}}$$

$$\frac{x}{4} = \frac{-y}{0} = \frac{z}{-1}$$

Therefore the eigen vectors corresponding to $\lambda = 1$ are given by

$$X = k \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

where $k \in R - 0$
Finding eigen vectors for the eigen value $\lambda = 2$
Corresponding to $\lambda = 2$ we have the following matrix equation

$$(A - (2)I)X = O$$

$$\begin{bmatrix} -4 & -8 & -12 \\ 1 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results in the following system of linear equations

$$-4\,x - 8\,y - 12\,z = 0 \quad \text{- - - (1)}$$
$$x + 2\,y + 4\,z = 0 \quad \text{- - - (2)}$$
$$-z = 0 \quad \text{- - - (3)}$$

No two equations are mutually linearly dependent Solving equations (1) and (2) using Cramers rule

$$\frac{x}{\begin{vmatrix} -8 & -12 \\ 2 & 4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -4 & -12 \\ 1 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -4 & -8 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x}{2} = \frac{-y}{-1} = \frac{z}{0}$$

Therefore the eigen vectors corresponding to $\lambda = 2$ are given by

$$X = k \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

where $k \in R - 0$

<u>Finding eigen vectors for the eigen value $\lambda = 0$</u>

Corresponding to $\lambda = 0$ we have the following matrix equation

$$(A - (0)I)X = O$$

$$\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results in the following system of linear equations

$$-2\,x - 8\,y - 12\,z = 0 \quad \text{- - - (1)}$$
$$x + 4\,y + 4\,z = 0 \quad \text{- - - (2)}$$
$$z = 0 \quad \text{- - - (3)}$$

No two equations are mutually linearly dependent Solving equations (1) and (2) using Cramers rule

$$\frac{x}{\begin{vmatrix} -8 & -12 \\ 4 & 4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & -12 \\ 1 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -8 \\ 1 & 4 \end{vmatrix}}$$

$$\frac{x}{4} = \frac{-y}{-1} = \frac{z}{0}$$

Therefore the eigen vectors corresponding to $\lambda = 0$ are given by

$$X = k \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

where $k \in R - 0$

[3] $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

For the given matrix A the characterstic equation is given by,

$$|A - xI| = 0$$

$$\begin{vmatrix} -x+3 & -1 & 1 \\ -1 & -x+5 & -1 \\ 1 & -1 & -x+3 \end{vmatrix} = 0$$

$$x^3 - 11\,x^2 + 36\,x - 36 = 0$$

$$(x-2)(x-3)(x-6) = 0$$

The eigen values of A are

$$\lambda = 2, 3, 6$$

Finding eigen vectors for the eigen value $\lambda = 2$

Corresponding to $\lambda = 2$ we have the following matrix equation

$$(A - (2)I)X = O$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results in the following system of linear equations

$$\begin{aligned} x - y + z = 0 \quad & \text{- - - (1)} \\ -x + 3\,y - z = 0 \quad & \text{- - - (2)} \\ x - y + z = 0 \quad & \text{- - - (3)} \end{aligned}$$

Here equations (1) and (3) are linealy dependent Solving equations (1) and (2) using Cramers rule

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$$

$$\frac{x}{1} = \frac{-y}{0} = \frac{z}{-1}$$

Therefore the eigen vectors corresponding to $\lambda = 2$ are given by

$$X = k \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

where $k \in R - 0$

Finding eigen vectors for the eigen value $\lambda = 3$

Corresponding to $\lambda = 3$ we have the following matrix equation

$$(A - (3)I)X = O$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results in the following system of linear equations

$$\begin{aligned} -y + z = 0 \quad & \text{- - - (1)} \\ -x + 2\,y - z = 0 \quad & \text{- - - (2)} \\ x - y = 0 \quad & \text{- - - (3)} \end{aligned}$$

No two equations are mutually linearly dependent Solving equations (1) and (2) using Cramers rule

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x}{1} = \frac{-y}{1} = \frac{z}{1}$$

Therefore the eigen vectors corresponding to $\lambda = 3$ are given by

$$X = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where $k \in R - 0$

<u>Finding eigen vectors for the eigen value $\lambda = 6$</u>

Corresponding to $\lambda = 6$ we have the following matrix equation

$$(A - (6)I)X = O$$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results in the following system of linear equations

$$\begin{aligned} -3\,x - y + z = 0 \quad & \text{- - - (1)} \\ -x - y - z = 0 \quad & \text{- - - (2)} \\ x - y - 3\,z = 0 \quad & \text{- - - (3)} \end{aligned}$$

No two equations are mutually linearly dependent Solving equations (1) and (2) using Cramers rule

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\frac{x}{1} = \frac{-y}{-2} = \frac{z}{1}$$

Therefore the eigen vectors corresponding to $\lambda = 6$ are given by

$$X = k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

where $k \in R - 0$

[4] $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

For the given matrix A the characterstic equation is given by,

$$|A - xI| = 0$$

$$\begin{vmatrix} -x-2 & 2 & -3 \\ 2 & -x+1 & -6 \\ -1 & -2 & -x \end{vmatrix} = 0$$

$$x^3 + x^2 - 21\,x - 45 = 0$$

$$(x+3)^2(x-5) = 0$$

The eigen values of A are

$$\lambda = 5, -3$$

<u>Finding eigen vectors for the eigen value $\lambda = 5$</u>
Corresponding to $\lambda = 5$ we have the following matrix equation

$$(A - (5)I)X = O$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results in the following system of linear equations

$$\begin{aligned} -7\,x + 2\,y - 3\,z = 0 \quad &\text{- - - (1)} \\ 2\,x - 4\,y - 6\,z = 0 \quad &\text{- - - (2)} \\ -x - 2\,y - 5\,z = 0 \quad &\text{- - - (3)} \end{aligned}$$

No two equations are mutually linearly dependent Solving equations (1) and (2) using Cramers rule

$$\frac{x}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x}{1} = \frac{-y}{2} = \frac{z}{-1}$$

Therefore the eigen vectors corresponding to $\lambda = 5$ are given by

$$X = k \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

where $k \in R - 0$

Finding eigen vectors for the eigen value $\lambda = -3$

Corresponding to $\lambda = -3$ we have the following matrix equation

$$(A - (-3)I)X = O$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results in the following system of linear equations

$$\begin{aligned} x + 2\,y - 3\,z &= 0 \quad \text{- - - (1)} \\ 2\,x + 4\,y - 6\,z &= 0 \quad \text{- - - (2)} \\ -x - 2\,y + 3\,z &= 0 \quad \text{- - - (3)} \end{aligned}$$

Here equations (1),(2) and (3) are linealy dependent So we consider any one of them. Let us consider

$$x + 2\,y - 3\,z = 0$$

$$x = -2\,y + 3\,z$$

Therefore the eigen vectors corresponding to $\lambda = -3$ are given by

$$X = \begin{bmatrix} -2\,y + 3\,z \\ y \\ z \end{bmatrix}$$

$$X = \begin{bmatrix} -2\,y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 3\,z \\ 0 \\ z \end{bmatrix}$$

$$X = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

where $y, z \in R - 0$

[5] $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

For the given matrix A the characterstic equation is given by,

$$|A - xI| = 0$$

$$\begin{vmatrix} -x & 1 & 1 \\ 1 & -x & 1 \\ 1 & 1 & -x \end{vmatrix} = 0$$

$$x^3 - 3\,x - 2 = 0$$

$$(x+1)^2(x-2) = 0$$

The eigen values of A are

$$\lambda = 2, -1$$

Finding eigen vectors for the eigen value $\lambda = 2$
Corresponding to $\lambda = 2$ we have the following matrix equation

$$(A - (2)I)X = O$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results in the following system of linear equations

$$\begin{aligned} -2\,x + y + z &= 0 \quad \text{- - - (1)} \\ x - 2\,y + z &= 0 \quad \text{- - - (2)} \\ x + y - 2\,z &= 0 \quad \text{- - - (3)} \end{aligned}$$

No two equations are mutually linearly dependent Solving equations (1) and (2) using Cramers rule

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x}{1} = \frac{-y}{1} = \frac{z}{1}$$

Therefore the eigen vectors corresponding to $\lambda = 2$ are given by

$$X = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where $k \in R - 0$

Finding eigen vectors for the eigen value $\lambda = -1$

Corresponding to $\lambda = -1$ we have the following matrix equation

$$(A - (-1)I)X = O$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results in the following system of linear equations

$$\begin{aligned} x + y + z &= 0 \quad \text{- - - (1)} \\ x + y + z &= 0 \quad \text{- - - (2)} \\ x + y + z &= 0 \quad \text{- - - (3)} \end{aligned}$$

Here equations (1),(2) and (3) are linealy dependent So we consider any one of them. Let us consider

$$x + y + z = 0$$

$$x = -y - z$$

Therefore the eigen vectors corresponding to $\lambda = -1$ are given by

$$X = \begin{bmatrix} -y - z \\ y \\ z \end{bmatrix}$$

$$X = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix}$$

$$X = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

where $y, z \in R - 0$