ALGEBRA US02CMTH21

MATRIX THEORY

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1 Special Types of Matrices

Zero Divisors:

If A and B are nonzero matrices such that $AB = 0$ then A and B are called zero divisors.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Theorem: The product of two matrices can be a zero matrix though none of them is a zero matrix.

Let $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ And $B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$

Then

$$AB = \begin{bmatrix} abc - abc & b^2c - b^2c & bc^2 - bc^2 \\ -a^2c + a^2c & -abc + abc & -ac^2 + ac^2 \\ ba^2 - a^2b & ab^2 - ab^2 & abc - abc \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

while $A \neq 0$ $B \neq 0$

Important Result:

$AB = AC$ does not imply $B = C$. i.e. cancellation laws does not hold for matrix multiplication.

e.g. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

And

$$AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

But $B \neq C$.

Idempotent Matrix:

A square matrix A is said to be idempotent if $A^2 = A$.

For example,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $A^2 = A$. Hence A is an idempotent matrix.

A square matrix A is said to be idempotent of period p if p is the least positive integer such that $A^{p+1} = A$.

For example,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And

$$A^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

So $A^{2+1} = A$. $\therefore$ A is an idempotent of period 2.

Nilpotent Matrix:

A square matrix A is said to be a nilpotent matrix if $A^k = 0$ where k is a positive integer.

If however k is the least integer for which $A^k = 0$ then k is called the index of nilpotent matrix A.

For example,

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ A is a nilpotent matrix index 2.

Involuntary Matrix:

A square matrix A is said to be an innvolutary matrix if $A^2 = I$.

Since $I^2 = I$ always, hence unit matrix I is involuntary matrix.

Let $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Then $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Hence A is also involuntary matrix.

Orthogonal Matrix:

A square matrix A is said to be orthogonal if $AA' = I = A'A$.

For example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then

$$A' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then

$$AA' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly we can show that $A'A = I$. Hence A is an orthogonal matrix.

Unitary Matrix:

A square matrix A is said to be unitary if $A^\theta A = I = AA^\theta$.

For example,

$$A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Then

$$A^\theta = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

Then

$$AA^\theta = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly we can show that $A^{\theta}A = I$. Hence A is a unitary matrix.

Theorem 1: If A and B are two idempotent matrices and if $AB = BA = 0$ then $A + B$ will also be an idempotent matrix.

Proof: Let A and B be two idempotent matrices.
Then $A^2 = A$ and $B^2 = B$, and suppose $AB = BA = 0$.
Now

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B)\\ &= A(A+B) + B(A+B) \quad (\because DistributiveLaw)\\ &= A^2 + AB + BA + B^2 \quad (\because DistributiveLaw)\\ &= A + B \quad (\because A^2 = A \quad and \quad B^2 = B)\end{aligned}$$

Hence $A + B$ is an idempotent matrix.

Theorem 2: If A and B are two idempotent matrices and if they commute then AB is idempotent matrix.

Proof: Let A and B be two idempotent matrices. Then $A^2 = A$ and $B^2 = B$.
Assume the A and B commutes. i.e. $AB = BA$

Now

$$
\begin{aligned}
(AB)^2 &= (AB)(AB) \\
&= A(BA)B \quad (\because AssociativeLaw) \\
&= A(AB)B \quad (\because AB = BA) \\
&= (AA)(BB) \quad (\because AssociativeLaw) \\
&= A^2B^2 \\
&= AB
\end{aligned}
$$

Hence AB is idempotent matrix.

Theorem 3: If A and B are two n-rowed orthogonal matrices then AB and BA are also orthogonal matrices.

Proof: Let A and B be two orthogonal matrices.
Then $AA' = A'A = I$ And $BB' = B'B = I$

Now

$$
\begin{aligned}
(AB)(AB)' &= (AB)(B'A') \\
&= A(BB')A' \quad (\because AssociativeLaw) \\
&= A(I)A' \\
&= (AI)A' \\
&= AA' \\
&= I
\end{aligned}
$$

Similarly

$$\begin{aligned}(AB)'(AB) &= (B'A')(AB)\\ &= B'(A'A)B \quad (\because AssociativeLaw)\\ &= B'(I)B\\ &= (B'I)B\\ &= B'B\\ &= I\end{aligned}$$

So we get $(AB)(AB)' = I = (AB)'(AB)$.

Hence AB is an orthogonal matrix.

Similarly we can show that BA is an orthogonal matrix.

Theorem 4: If A and B are two n-rowed unitary matrices then AB and BA are also unitary matrices.

Proof: Let A and B be two unitary matrices.

Then $AA^\theta = A^\theta A = I$ And $BB^\theta = B^\theta B = I$

Now

$$\begin{aligned}(AB)(AB)^\theta &= (AB)(B^\theta A^\theta)\\ &= A(BB^\theta)A^\theta \quad (\because AssociativeLaw)\\ &= A(I)A^\theta\\ &= (AI)A^\theta\\ &= AA^\theta\\ &= I\end{aligned}$$

Similarly

$$
\begin{aligned}
(AB)^\theta(AB) &= (B^\theta A^\theta)(AB) \\
&= B^\theta(A^\theta A)B \quad (\because AssociativeLaw) \\
&= B^\theta(I)B \\
&= (B^\theta I)B \\
&= B^\theta B \\
&= I
\end{aligned}
$$

So we get $(AB)(AB)^\theta = I = (AB)^\theta(AB)$.

Hence AB is an unitary matrix.

Similarly we can show that BA is an unitary matrix.

Example 1: Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is an idempotent matrix.

Solution: We have

$$
A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}
$$

Then

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= A$$

$\therefore A^2 = A.$

Hence A is idempotent matrix.

Example 2: Show that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of index 3.

Solution: Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

Then

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-3+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & -3 \\ -1 & -1 & -3 \end{bmatrix}$$

Then

$$A^3 = A^2 A$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & -3 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $A^3 = 0$.

Hence A is a nilpotent matrix of index 3.

Example 3: Prove that the matrix $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary.

Solution: Let $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

Then $A' = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$

Then $A^\theta = \overline{A'} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

Then

$$AA^{\theta} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1+(1-i)^2 & 1+i-(1+i) \\ (1-i)-(1+i) & 1-i^2+1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Thus $AA^{\theta} = I$. Similarly we get $A^{\theta}A = I$

Hence A is a unitary matrix.

Example 4: Show that $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involuntary.

Solution: We have $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Then

$$A^2A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 - 24 + 0 & 40 - 40 + 0 & 0 + 0 + 0 \\ -15 + 15 + 0 & -24 + 25 + 0 & 0 + 0 + 0 \\ -5 + 6 - 1 & -8 + 10 - 2 & 0 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Thus $A^2 = I$. Hence A is involuntary matrix.

Example 5: Determine the values of α,β and γ when $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal.

Solution: We have $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$

Then $A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & \beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$.

If A is orthogonal then

$$AA' = I$$

$$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & \beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore 4\beta^2 + \gamma^2 = 1 \quad 2\beta^2 - \gamma^2 = 0 \quad \alpha^2 + \beta^2 + \gamma^2 = 1$ and $\alpha^2 - \beta^2 - \gamma^2 = 0$

Then $6\beta^2 = 1 \Rightarrow \beta = \frac{\pm 1}{\sqrt{6}}$

Also $\gamma^2 = 2\beta^2 \Rightarrow \gamma^3 = \frac{1}{3} \Rightarrow \gamma = \frac{\pm 1}{\sqrt{3}}$

Also from $\alpha^2 + \beta^2 + \gamma^2 = 1$ and $\alpha^2 - \beta^2 - \gamma^2 = 0$ we get $2\alpha^2 = 1$. Then $\alpha = \frac{\pm 1}{\sqrt{2}}$

Example 6: If A is an idempotent matrix and $A + B = I$ then show that B is also an idempotent matrix and $AB = BA = 0$.

Solution: Let A be an idempotent matrix and suppose $A + B = I$.

Then

$$B = I - A$$

Then

$$\begin{aligned} B^2 &= (I - A)^2 \\ &= (I - A)(I - A) \\ &= I(I - A) - A(I - A) \quad (\because DistributiveLaw) \\ &= I - A - A + A^2 \quad (\because DistributiveLaw) \\ &= I - A - A + A \quad (\because A^2 = A) \\ &= I - A \\ &= B \end{aligned}$$

Thus $B^2 = B$. Hence B is an idempotent matrix.

Also $AB = A(I - A) = A - A^2 = A - A = 0$

And $BA = (I - A)A = A - A^2 = A - A = 0$

Thus $AB = BA = 0$

Example 7: If A is a real skew-symmetric matrix such that $A^2 + I = 0$ then show that A is orthogonal and is of even order.

Solution: Let A be a skew-symmetric matrix such that $A^2 + I = 0$.

Then $A' = -A$.

Also $A^2 + I = 0 \Rightarrow A^2 = -I$

Now $AA' = A(-A) = -A^2 = -(-I) = I$

Similarly $A'A = (-A)A = -A^2 = -(-I) = I$.

Thus $AA' = A'A = I$. Hence A ia an orthogonal matrix.

Now suppose A is an n $\times$ n matrix.

Since $A' = (-A) = (-1)A$

Then $|A'| = |(-1)A|$

$\Rightarrow |A| = (-1)^n|A| \quad (\because |A'| = |A| \text{ and } |kA| = k^n|A|)$

$\Rightarrow (1 - (-1)^n)|A| = 0$

Then either $(1 - (-1)^n) = 0$ or $|A| = 0$.

But since A is orthogonal $|A| \neq 0$.

Then $(1 - (-1)^n) = 0$ which is possible only if n is even.

Hence A is an orthogonal matrix of even order.

2 Adjoint of a square matrix

Definition: Let A be a square matrix. The matrix $[A_{ij}]$ where A_{ij} denotes the cofactor of a_{ji} in the determinant of A is called the adjoint of A and is denoted by the symbol $adj\ A$.

(The cofactor of $a_{ij} = (-1)^{i+j}Q_{ij}$ where Q_{ij} is the determinant of the matrix obtained by removing the i^{th} row and j^{th} column.)

So the adjoint of A is the transpose of the matrix formed by the cofactors of A.

If $A = \begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{bmatrix}$

Then $adjA = \begin{bmatrix} A_{11} & A_{21} & \dots & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & \dots & A_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ A_{1i} & A_{2i} & \dots & \dots & A_{ni} \\ \dots & \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & \dots & A_{nn} \end{bmatrix}$

where the elements of each column(row) of adj A are the cofactors of the corresponding elements of the corresponding row(column) of A in $|A|$.

For example let $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 2 & 0 & -4 \end{bmatrix}$

Then $$A' = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 3 & -1 & -4 \end{bmatrix}$$

Then $$adjA = \begin{bmatrix} -4 & 4 & -4 \\ -2 & -10 & 1 \\ -2 & 2 & 1 \end{bmatrix}$$

An Important Relation Between A and adj A:

Theorem 5 : $A(adjA) = |A|I = (adjA)A.$

Proof: The $(i, j)^{th}$ entry of A(adj A)

$$= a_{i1}A_{1j} + a_{i2}A_{2j} + \cdots + a_{in}A_{nj}$$

$$= 0 \quad if \quad i \neq j$$

$$|A| \quad if \quad i = j$$

Thus $A(adjA) = Diag[|A|, |A|, \ldots, |A|]$

Similarly $(adjA)A = Diag[|A|, |A|, \ldots, |A|]$

Hence $A(adjA) = |A|I = (adjA)A$

Corollary: If $|A| \neq 0$ then,

$$A(\tfrac{1}{|A|}adjA) = I = (\tfrac{1}{|A|}adjA)A$$

3 Inverse of a Matrix:

Definition: If A be any given matrix then a matrix B if it exists such that $AB = I = BA$ is called inverse of A,where I being the unit matrix.

For the product AB and BA both to be defined and equal it is necessary that A and B are square matrices of same order. Thus nonsquare matrices can not possess inverses.

Remark: The inverse of a matrix if it exists must be unique.

Let A be given square matrix. If possible assume that B and C are two inverses of A.
Then $AB = BA = I$ and $AC = CA = I$.
Now $B = B(I) = B(AC) = (BA)C = IC = C$.
$\therefore B = C$

Existence of Inverse:

Theorem 6: A necessary and sufficient condition for a square matrix A to possess inverse is that $|A| \neq 0$

Proof: Let A be any given square matrix.

- The condition $|A| \neq 0$ is necessary.
  Suppose that A has an inverse.
  Then there exists a matrix B such that

$$AB = I = BA.$$

$$\therefore |AB| = |I|$$

$$\therefore |A||B| = 1$$

$$\therefore |A| \neq 0$$

- The condition $|A| \neq 0$ is sufficient.

  Suppose $|A| \neq 0$.

  Let $B = \frac{1}{|A|}(adjA)$.

  Then

$$\begin{aligned} AB &= A(\frac{1}{|A|}(adjA)) \\ &= \frac{1}{|A|}(A(adjA)) \\ &= \frac{1}{|A|}(|A|I) \quad (\because \quad by \quad Theorem\, 5 \quad A(adjA) = |A|I = (adjA)A) \\ &= I \end{aligned}$$

Similarly $BA = I$.

Thus B is the inverse of A.

Non-singular and Singular Matrices:

A square matrix A is said to be non-singular matrix if $|A| \neq 0$ and is said to be singular matrix if $|A| = 0$.

Thus only non-singular matrices possess inverses.

NOTE: It may be seen that if the elements of a non-singular matrix A are members of F then the elemnts of its inverse A^{-1} are also members of F where F can be either of $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Reversal Law for the Inverse of a Product:

Theorem 7: If A and B be two non-singular matrices of the same order, then the product AB is non-singular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

i.e inverse of a product is the product of the inverses taken in reverse order.

Proof: Let A and B be two non-singular matrices.

Then A^{-1} and B^{-1} exist. Now

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \quad (\because AssociativeLaw)\\ &= AIA^{-1}\\ &= (AI)A^{-1}\\ &= AA^{-1}\\ &= I\end{aligned}$$

Similarly $(B^{-1}A^{-1})(AB) = I$.

Thus $B^{-1}A^{-1}$ is the inverse of AB. Hence inverse of AB exists.

Then by theorem 6 $|AB| \neq 0$.

Then we can say that AB is non-singular matrix.

Generalization:

By successive applications of the Associative Law, we can easily generalize the above result and show that if $A_1, A_2, \ldots, A_k$ are non-singular matrices of the same order, then

$$(A_1 A_2 \dots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \dots A_1^{-1}$$

Negative Integral Powers of a Non-Singular Matrix:

If A is a non-singular matrix and k a positive integer, then by definition, we write

$$A^{-k} = (A^k)^{-1}$$

so that A^{-k} is the inverse of A^k. Also we agree to write

$$A^0 = I$$

By generalized reversal law we have

$$\begin{aligned}(A^k)^{-1} &= (AA \dots A)^{-1} \\ &= (A^{-1}A^{-1} \dots A^{-1} \\ &= (A^{-1})^k \\ \therefore (A^k)^{-1} &= (A^{-1})^k\end{aligned}$$

It may now be easily shown that

$$A^m A^n = A^{m+n}$$

And

$$(A^m)^n = A^{mn}$$

where m and n can be either of positive integer, negative integer or zero and A should be a non-singular matrix.

Theorem 8: The operations of transposing and inverting are commutative.

i.e.

$$(A')^{-1} = (A^{-1})'$$

Proof: Let A be a non-singular matrix.

Then $AA^{-1} = I = A^{-1}A$

Taking transpose, we obtain

$$(AA^{-1})' = I' = (A^{-1}A)'$$

$$\therefore (A^{-1})'A' = I = A'(A^{-1})' \quad (\because \quad (AB)' = B'A')$$

$\therefore (A^{-1})'$ is the inverse of A'.

$$\therefore (A')^{-1} = (A^{-1})'$$

NOTE: Similarly by taking conjugate transpose we can get

$$(A^{\theta})^{-1} = (A^{-1})^{\theta}$$

HOMEWORK: If I_n be a unit matrix of order n show that

$$adj I_n = I_n$$

Example 8: If A is a square matrix then show that $adj A' = (adj A)'$.

Solution: Let A be a square matrix of order n.

Then both $adj A'$ and $(adj A)'$ are square matrices of order n.

Now

the $(i,j)^{th}$ element of $(adj A)'$ = the $(j,i)^{th}$ element of adj A

= the cofactor of $(i,j)^{th}$ element in $|A|$

Also

the $(i,j)^{th}$ element of $adj A'$ = the cofactor of $(j,i)^{th}$ element in $|A'|$

= the cofactor of (i,j)th element in $|A|$

Thus $(adj A)' = adj A'$.

HOMEWORK: If A is a symmetric matrix then show that adj A is also symmetric.

Example 9: If A and B are square matrices of same order then

$$adj\ (AB) = adj\ (B)\ adj\ (A)$$

(provided that AB is non-singular)

Solution: We have

$$AB(adj\ (AB)) = |AB|I = (adj\ (AB))AB \text{ ......(1)}$$

Also

$$\begin{aligned} AB(adj\ (B)\ adj\ (A)) &= A(B\ adj\ (B))adj\ (A) \quad (\because Associatve\ Law) \\ &= A(|B|I)adj\ (A) \quad (\because B(adj\ B) = |B|I = (adj\ B)B) \\ &= |B|(A(adj\ A)) \\ &= |B||A|I \\ &= |A||B|I \text{........(2)} \end{aligned}$$

Then from (1) and (2) we get

$$AB(adj\ (AB)) = AB(adj\ (B)\ adj\ (A))$$

Multiplying both sides by $(AB)^{-1}$ we get,

$$adj\ (AB) = adj\ (B)\ adj(A)$$

Example 10: If A be an $n \times n$ matrix then prove that

$$|adj\ A| = |A|^{n-1}$$

(Provided that $|A| \neq 0$)

Solution: We know that,

$$A\ adj\ A = |A|\ I_n$$

$$\therefore\ |A\ adj\ A| = ||A|\ I_n|$$

$$\therefore\ |A|\ |adj\ A| = |A|^n\ |I_n| \quad (\because\ |AB| = |A||B|\ and\ |kA| = k^n|A|)$$

$$\therefore\ |A|\ |adj\ A| = |A|^n$$

$$\therefore\ |adj\ A| = |A|^{n-1} \quad if\ |A| \neq 0$$

Example 10: Show that if A and B are symmetric matrices and commute then

(a) $A^{-1}B$ (b) AB^{-1} and (c) $A^{-1}B^{-1}$

are symmetric matrices.

Solution: Let A and B be two symmetric matrices.
Then $A' = A$ and $B' = B$
Assume that A and B commute.

$$\text{i.e. } AB = BA$$

(a)

$$\begin{aligned}(A^{-1}B)' &= (B')((A^{-1})') \\ &= B'((A')^{-1}) \quad (\because \quad (A^{-1})' = (A')^{-1}) \\ &= BA^{-1} \quad (\because \quad A' = A \quad and \quad B' = B)\text{.............}(1)\end{aligned}$$

Since

$$\begin{aligned}AB &= BA \\ \Rightarrow A^{-1}(AB)A^{-1} &= A^{-1}(BA)A^{-1} \\ \Rightarrow BA^{-1} &= A^{-1}B\text{................}(2)\end{aligned}$$

Using (2) in (1) we get

$$(A^{-1}B)' = (A^{-1}B)$$

hence $A^{-1}B$ is symmetric matrix.

(b) HOMEWORK

(c)

$$\begin{aligned}(A^{-1}B^{-1})' &= (B^{-1})'(A^{-1})' \\ &= (B')^{-1}(A')^{-1} \quad (\because (A^{-1})' = (A')-1) \\ &= B^{-1}A^{-1}\text{....................}(3)\end{aligned}$$

Since

$$AB = BA$$
$$\Rightarrow (AB)^{-1} = (BA)^{-1}$$
$$\Rightarrow B^{-1}A^{-1} = A^{-1}B^{-1} \text{..................(4)}$$

Using (4) in (3) we get

$$(A^{-1}B^{-1})' = A^{-1}B^{-1}$$

Hence $A^{-1}B^{-1}$ is a symmetric matrix.

Example 11: If $adj\ B = A$ and P and Q are two unimodular matrices then prove that

$$adj\ (Q^{-1}BP^{-1}) = PAQ$$

Solution: Since P and Q are unimodular,

$$|P| = |Q| = 1$$

Now

$$Q^{-1}BP^{-1} = (Q^{-1}B)P^{-1}$$

Then

$$\begin{aligned} adj\ (Q^{-1}BP^{-1}) &= adj\ ((Q^{-1}B)P^{-1}) \\ &= adj\ (P^{-1})adj\ (Q^{-1}B) \quad (\because \quad adj\ (AB) = adj\ B\ adj\ A) \\ &= adj\ (P^{-1})adj\ B\ adj\ (Q^{-1}) \end{aligned}$$

Thus $adj\ (Q^{-1}BP^{-1}) = adj\ (P^{-1})adj\ B\ adj\ (Q^{-1})$ ....(1)

We know that

$$P^{-1} adj\ (P^{-1}) = |P^{-1}|\ I \quad ....(2)$$

$$\therefore\ adj\ (P^{-1}) = P|P^{-1}|I$$
$$= P\frac{1}{|P|}I$$
$$= P \quad (\because \quad |P| = 1)$$

Similarly

$$adj\ (Q^{-1}) = Q \quad ....(3)$$

Using (2) and (3) in (1) we get

$$adj\ (Q^{-1}BP^{-1}) = P\ adj\ B\ Q = PAQ \quad (\because adj\ B = A)$$

NOTATIONS:

We consider F as either of $\mathbb{Q}$,$\mathbb{R}$ or $\mathbb{C}$.

Let $m, n \in \mathbb{N}$. Then

$$M_{m\times n} = \text{Set of all } m \times n \text{ matrices with elements from F}$$

And

$$M_{n\times n} = \text{Set of all } n \times n \text{ square matrices with elements from F}$$

Note that $M_{m\times n}$ is closed under matrix addition,while $M_{n\times n}$ is closed under matrix addition as well as matrix multiplication.

4 Left and Right Zero Divisors:

If $AB = 0$ and if $A \neq 0$ then A is called a left zero divisor, and if $AB = 0$ and if $B \neq 0$ then B is called the right zero divisor.

5 Trace of Matrix:

Definition: Let A be a square matrix of order n. The sum of the elements of A lying along the principal diagonal is called the trace of A. We shall write trace of A as *tr. A*.

Thus if $A = [a_{ij}]_{n\times n}$ then $tr.\ A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$

Theorem 9: Let A and B be two square matrices of order n and λ be a scalar. Then

$$\text{(i) } tr.\ (\lambda\ A) = \lambda(tr.\ A) \quad \text{(ii) } tr.\ (A+B) = tr.\ (A) + tr.\ (B)$$
$$\text{(iii) } tr.\ (AB) = tr.\ (BA)$$

Solution: Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$

(i) We have

$$\lambda\ A = [\lambda\ a_{ij}]_{n\times n}$$

$$\begin{aligned} \therefore \quad tr.\ (\lambda\ A) &= \sum_{i=1}^{n} \lambda a_{ii} \\ &= \lambda \sum_{i=1}^{n} a_{ii} \\ &= \lambda\ (tr.\ A) \end{aligned}$$

(ii) We have

$$A + B = [(a_{ij} + b_{ij})]_{n\times n}$$

$$\therefore tr.\ (A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii})$$

$$= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$$

$$= tr.\ A + tr.\ B$$

(iii) We have

$$AB = [c_{ij}]_{n\times n} \quad \text{where} \quad c_{ij} = \textstyle\sum_{k=1}^{n} a_{ik}b_{kj}$$

And

$$BA = [d_{ij}]_{n\times n} \quad \text{where} \quad d_{ij} = \textstyle\sum_{k=1}^{n} b_{ik}a_{kj}$$

Now,

$$tr.\ (AB) = \sum_{i=0}^{n} c_{ii}$$

$$= \sum_{i=0}^{n}(\sum_{k=1}^{n} a_{ik}b_{ki})$$

Interchanging the order of summation in the last sum

$$tr.\ (AB) = \sum_{k=1}^{n}(\sum_{i=1}^{n} b_{ki}a_{ik})$$

$$= \sum_{k=1}^{n} d_{kk}$$

$$= d_{11} + d_{22} + \cdots + d_{nn}$$

$$= tr.\ (BA)$$

6 Elementary Transformation of a Matrix:

The following transformation, three of which refer to rows and three of which refer to columns are known as Elementary Transformation.

I. Interchange of two rows(columns).

II. The multiplication of a row (column) by a non-zero number.

III. The addition to the elments of a row(column), the corresponding elements of a row(column) multiplied by any number.

Symbols to be employed for the Transformation:

1. R_{ij} - interchanging i^{th} and j^{th} row.
2. $R_i(c)$ - multiplication of th i^{th} row by $c \neq 0$.
3. $R_{ij}(k)$- addition to the i^{th} row , the product of the j^{th} row by k.

The corresponding column transformation will be denoted by writing C, in place of R. i.e. by C_{ij} ,$C_i(c)$, $C_{ij}(k)$ respectively.

7 Reduced Row-Echelon Form:

A matrix is said to be in *reduced row echelon* form if all the following conditions are satisfied:

1. The first non-zero element in each row is 1.(This 1 is called a leading one).

2. Each successive row has (from upper to lower) the leading one in a column farther to the right.

(Each leading one of a row should be to the right of the leading one in the previous row)

3. The leading one in each row is the only non-zero entry in the column containing this leading one.

4. The zero rows (row in which each entry is zero) are the final rows of the matrix.

Row-Echelon Form:

Definition: A matrix satisfying conditions 1, 2 and 4 is said to be in *Row-Echelon Form.*

Thus every matrix in Reduced raw-echelon form is also in Raw-echelon form, but the converse may not necessarily be true.

Examples: Consider the following matrices.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A is not in Reduced row echelon form and also not in row echelon form because zero row is placed before non zero row. (Condition 4 not satisfied.)

B is not in Reduced row echelon form because in third row entry above leading one is nonzero,(Condition 3 not satisfied) but B is in row echelon form.

C is not is reduced row echelon form and also not in row echelon form because the leading one in first row is lying in third column and the leading one in second row lies in first column.(Condition 2 not satisfied.)
D is in reduced row echelon form as well as in row echelon form.

$$E = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here E and F both are in reduced row echelon form as well as in row echelon form.

Remark:
If a matrix is in reduced row echelon form then it is also in row echelon form.

Example 12: Convert $A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$ into its equivalent row-echelon form.

Solution: We have

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$R_2 \to (-1)R_2$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$R_1 \to R_1 + 2R_2$,$R_3 \to R_3 - 5R_2$, $R_4 \to R_4 - 4R_2$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 6 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 6 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to (\frac{1}{6})R_3$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 6 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_3,\ R_1 \to R_1 - 6R_3$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the reduced row echelon form.

8 Rank of a matrix (Using Reduced Row Echelon Form:

Definition: Let A be a $m \times n$ matrix, then the rank of the matrix A is the number of non-zero rows in the reduced row echelon form of A and is denoted by $rank\ (A)$ or $\rho(A)$.

9 Nullity of a Matrix:

If A is square matrix of order n then $n - \rho(A)$ is called the nullity of the matrix and it is denoted by $N(A)$.

Example 13: Obtain the reduced row echelon form of the matrix $A = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 8 \end{bmatrix}$ and hence find the rank of the matrix A.

Solution: We have

$$A = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 8 \end{bmatrix} \quad R_{21}(-1) \quad R_{31}(-2) \quad R_{41}(-3)$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & -1 & -1 & 1 \\ 0 & -2 & -1 & 0 \\ 0 & -2 & -2 & 2 \end{bmatrix} \quad R_2(-1)$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -1 & 0 \\ 0 & -2 & -2 & 2 \end{bmatrix} \quad R_{12}(-3) \quad R_{32}(2) \quad R_{42}(2)$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_{23}(-1) \quad R_{13}(1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is the reduced row echelon form and $\rho(A) = 3$.

Example 14: Determine rank of A if

$$A = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$$

Solution: We have,

$$A = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix} \quad R_{12}(-1)$$

$$\sim \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix} \quad R_1(-1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix} \quad R_{21}(-1) \quad R_{31}(-5) \quad R_{41}(-10) \quad R_{51}(-15)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \quad R_{12}(-1) \quad R_{32}(-1) \quad R_{42}(-1) \quad R_{52}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is the reduced row echelon form and hence $\rho(A) = 2$.

Example 15: Find the rank of the matrix $A = \begin{bmatrix} 8 & 0 & 0 & 16 \\ 0 & 0 & 0 & 6 \\ 0 & 9 & 9 & 9 \end{bmatrix}$

Solution: We have

$$A = \begin{bmatrix} 8 & 0 & 0 & 16 \\ 0 & 0 & 0 & 6 \\ 0 & 9 & 9 & 9 \end{bmatrix} \quad R_1(\frac{1}{8})$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 6 \\ 0 & 9 & 9 & 9 \end{bmatrix} \quad R_{32}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad R_2(\frac{1}{9})$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad R_3(\frac{1}{6})$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_{13}(-1) \quad R_{23}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Which is the reduced row echelon form and hence $\rho(A) = 3$.

10 Inverse of a matrix (Using Gauss-Jordan Method):

Working Rule:

Let A be an $n \times n$ square matrix.

Step 1: Augment the matrix A in the form $[A|I_n]$, which is of order $n \times 2n$, in which the first n columns are that of A and the remaining n columns are that of I_n.

Step 2: Obtain reduced row echelon form of $[A|I_n]$ by using elementary row transformation.

Step 3: If the first n columns of $[A|I_n]$ are converted into I_n then A is non-singular and the last n columns of the augmented matrix in the reduced row echelon form is the inverse of A. That is $[A|I_n]$ is reduced to the form $[I_n|A^{-1}]$

Step 4: If the first n columns of $[A|I_n]$ cannot be converted into I_n then A is singular and stop the procedure.

Example 16: Find the inverse of $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$.

Solution: We have $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

Then

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 3 & -3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_{12}(-1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_{21}(-2)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -3 & 4 & -2 & 3 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_{23}(-4)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_{32}(1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right] = [I_3|A^{-1}]$$

Hence

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Example 17: Find A^{-1} using row operations if $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Solution: We have

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_{21}(1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_{32}(-1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \quad R_3(\frac{-1}{2})$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right] \quad R_{13}(-1) \quad R_{23}(-2)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right] = [I_3|A^{-1}]$$

Hence

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$$

Example 18: By using Gauss Jordan method find the inverse of the matrix $A = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{-4}{5} & \frac{1}{10} \end{bmatrix}$.

Solution: We have $A = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{-4}{5} & \frac{1}{10} \end{bmatrix}$.

Then

$$[A|I_3] = \left[\begin{array}{ccc|ccc} \frac{1}{5} & \frac{1}{5} & \frac{-2}{5} & 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & \frac{-4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right] \quad R_1(5)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & \frac{-4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right] \quad R_{21}(\frac{-1}{5}) \quad R_{31}(\frac{-1}{5})$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \\ 0 & -1 & \frac{1}{2} & -1 & 0 & 1 \end{array}\right] \quad R_{23}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & -1 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \end{array}\right] \quad R_2(-1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \end{array}\right] \quad R_{12}(-1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{-3}{2} & 4 & 0 & 1 \\ 0 & 1 & \frac{-1}{2} & 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \end{array}\right] \quad R_{13}(3) \quad R_{23}(1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \end{array}\right] \quad R_3(2)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right] = [I_3|A^{-1}]$$

Hence

$$A^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}$$

Example 19: Find the inverse of the matrix A by Gauss Jordan Method where

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution: We have

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Then

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_{12}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_{31}(-3)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right] \quad R_{12}(-2) \quad R_{32}(5)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right] \quad R_3(\frac{1}{2})$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{array}\right] \quad R_{13}(1) \quad R_{23}(-2)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{array}\right] = [I_3|A^{-1}]$$

Thus

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

11 Solution of a System Using Inverse of a Matrix:

Theorem 10: Let $AX = B$ be a system of linear equations where A is a square coefficient matrix.

I If A is a non-singular matrix then the system has a unique solution $X = A^{-1}B$.

II If A is singular then the system has no solutions or infinitely many solutions.

Proof:

I Let A be a non-singular matrix. Then A^{-1} exists.

First we show that $X = A^{-1}B$ is a solution of $AX = B$. Now

$$A(A^{-1}B) = (AA^{-1})B = IB = B.$$

$$\therefore X = A^{-1}B \text{ is a solution of } AX = B.$$

Next we show that this solution is unique. Suppose Y is any other solution of the system.

Then we have $AY = B$.

Then $A^{-1}(AY) = A^{-1}B$.

$\therefore Y = A^{-1}B = X$.

Thus $A^{-1}B$ is a unique solution of $AX = B$.

II If A is singular then $\rho(A) < n$.

Then the reduced row echelon form must have some zero rows.

Then there must be at least an independent variables which assumes any arbitrary value. Therefore there exists infinite number of solution or no solution exists.

Example 20: Solve the following system of equations by inverse method.

$$x - 4y + 5z = 8$$
$$3x + 7y - z = 3$$
$$x + 15y - 11z = 14$$

Solution: The system in matrix form is given by

$$\begin{bmatrix} 1 & -4 & 5 \\ 3 & 7 & -1 \\ 1 & 15 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 14 \end{bmatrix}$$

Then

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 3 & 7 & -1 \\ 1 & 15 & -11 \end{bmatrix}$$

$$\therefore [A|I_3] = \left[\begin{array}{ccc|ccc} 1 & -4 & 5 & 1 & 0 & 0 \\ 3 & 7 & -1 & 0 & 1 & 0 \\ 1 & 15 & -11 & 0 & 0 & 1 \end{array}\right] \quad R_{21}(-3) \quad R_{31}(-1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -4 & 5 & 1 & 0 & 0 \\ 0 & 19 & -16 & -3 & 1 & 0 \\ 0 & 19 & -16 & -1 & 0 & 1 \end{array}\right] \quad R_{32}(-1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -4 & 5 & 1 & 0 & 0 \\ 0 & 19 & -16 & -3 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 \end{array}\right] \quad R_3(\frac{1}{19})$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -4 & 5 & 1 & 0 & 0 \\ 0 & 1 & \frac{-16}{19} & \frac{-3}{19} & \frac{1}{19} & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 \end{array}\right] \quad R_{12}(4)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{31}{19} & \frac{7}{19} & \frac{4}{19} & 0 \\ 0 & 1 & \frac{-16}{19} & \frac{-3}{19} & \frac{1}{19} & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 \end{array}\right]$$

The third row is a zero row . Hence A^{-1} does not exist.

Thus the system has no solutions or infinite number of solution.

Example 21: Solve the following system of equations by inverse method

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

Solution: The system in matrix form is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$\therefore [A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right] \quad R_{21}(-1) \quad R_{31}(-1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 8 & -1 & 0 & 1 \end{array}\right] \quad R_{12}(-1) \quad R_{32}(-3)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 & -3 & 1 \end{array}\right] \quad R_3(\frac{1}{2})$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & \frac{-3}{2} & \frac{1}{2} \end{array}\right] \quad R_{13}(1) \quad R_{23}(-2)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & \frac{-5}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & \frac{-3}{2} & \frac{1}{2} \end{array}\right] \quad R_{13}(1) \quad R_{23}(-2)$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & \frac{-5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

Then

$$X = A^{-1}B = \begin{bmatrix} 3 & \frac{-5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Thus we get $x = 2$,$y = 1$,$z = 0$.

12 Rank of a Matrix (Minor Form):

Minor of a matrix:

Let A be any $m \times n$ matrix and let t be any natural number such that

$$t \leq min(m, n)$$

Now we delete (m-t) rows and (n-t) columns of the matrix, then the remaining elements form a t-rowed square submatrix whose determinant is a minor of the matrix A of order t.

Rank of a matrix:

For a matrix A a number r with the followinng two properties is called the ranl of the matrix:

(i) There is atleast one non-zero minor of order r

(ii) Every minor of order (r+1) is zero.

Briefly we may say that the rank of a matrix is the largest order of a non-zero minor of a matrix. Rank of a matrix is denoted by $\rho(A)$.

Remarks:

(1) $\rho(A) \leq n$.

(2) Rank of every non-zero matrix is at least 1 and the rank of zero matrix is zero.

(3) $\rho(A') = \rho(A)$ and $\rho(A^\theta) = \rho(A)$

(4) Rank of a non-singular matrix of order n is n.

Example 22: Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$

Solution: We have $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$

Then $|A| = 0$.

Now,

$$\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -7.$$

Thus $\rho(A) = 2$.

Invariance of Rank Through Elementary Transformation:

Theorem 11: Elementary transformations of a matrix do not alter its rank.

Proof: Assume it without proof.

Corollary: The elementary column transformation do not alter its rank, this is because the rank of A' and A are same.

Reduction to Normal Form:

Theorem 11: Every non-zero matrix of rank r can, by a sequence of elementary transformations be reduced to the form

$$\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

where I_r being the r-rowed unit matrix.

The form obtained here is called the Normal form.

Proof: Assume without proof.

NOTES:

1. As the rank of the matrix does not alter due to elementary transformations, the rank of the normal form will be same as the rank of a given matrix A.

2. In evaluation of the rank of a matrix by method of elementary transformations, if certain rows and columns are reduced to zero entirely, we can remove them without affecting the rank of the matrix. This method is known as Sweep Out or Pivotal Method.

Example 23: Reduce the matrix A to its normal form where $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Hence find its rank.

Solution: We have

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \quad R_{12}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad R_{31}(-3) \quad R_{41}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_{32}(-1) \quad R_{42}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_{32}(-1) \quad C_{42}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_{31}(-1) \quad C_{41}(-1)$$

$$\sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$

This is the normal form of A and $\rho(A) = 2$.

13 Elementary Matrices:

Definition: A matrix obtained from a unit matrix by subjecting it to any of the elementary transformation is called an elementary matrix.

Symbols For Elementary Matrices:

I. E_{ij} - the matrix obtained by interchanging i^{th} row and j^{th} row. It may easily be seen that the matrices obtained by changing i^{th} row and j^{th} row or by interchanging i^{th} and j^{th} column are the same.

II. (a) $E_i(c)$ will denote the matrix obtained by multiplying the i^{th} row by c.

(b) $E_i(c)$ will also denote the matrix obtained by multiplying the i^{th} column with c.

III. (a) $E_{ij}(k)$ will denote the matrix , obtained by adding to the elements of the i^{th} row of the unit matrix , the products by k of the corresponding elements of j^{th} row.

(b) $E'_{ij}(k)$ which is the transpose of $E_{ij}(k)$ will denote the matrix, obtained by adding to the elements of the i^{th} column , the products by k of the corresponding elements of the j^{th}.

Determinants of Elementary Matrices:

It is easy to see that

$$|E_{ij}| = -1, \quad |E_i(c)| = c \neq 0, \quad |E_{ij}(k)| = |E'_{ij}(k)| = 1$$

Elementary Transformation And Elementary Matrices:

Lemma: Every elementary row(column) transformation of a product of two matrices can be effected by subjecting the pre-factor (post-factor) to the same row (column) transformation.

Theorem 12: Every elementary row(column) transformation of a matrix can be brought about by pre-multiplication(post-multiplication) with the corresponding elementary matrix.

Proof: Assume it without proof.

Example 24: For $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ find non-singular matrices P and Q such that PAQ is in a normal form.

Solution: We write $A = IAI$

Every elementary row(column) transformation of the product will be affected by subjecting the pre-factor(post-factor) of A to the same.

We have

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $C_{21}(-1)$ and $C_{31}(-2)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $R_{21}(-1)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $C_{32}(-1)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $R_{32}(1)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus we have the required normal form. We have

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

14 Equivalence of Matrices:

Definition: Let $A \in M_{m\times n}(F)$ and $B \in M_{m\times n}(F)$. A is said to be equivalent to B, if there exists two non-singular matrices P and Q whose elements are members of F such that

$$A = PBQ$$

The following properties of this relation are fundamental.

I Reflexivity: Every matrix A is equivalent to itself, for we have

$$A = IAI \quad \text{so that } P = I \text{ and } Q = I.$$

II Symmetry: If A is equivalent to B over F then B is also equivalent to A over F, for

$$A = PBQ \Rightarrow B = P^{-1}AQ^{-1}$$

where P^{-1} and Q^{-1} are non-singular matrices over F.

III Transitivity: If A is equivalent to B over F and B is equivalent yo C over F then A is also equivalent to C over F for

$$A = PBQ \text{ and } B = LCM$$

$$\Rightarrow A = PLCMQ = (PL)C(MQ)$$

where PL, MQ being the product of non-singular matrices over F are themselves non-singular over F.

Thus the relation of being equivalent is reflexive, Symmetric and transitive.

Result: Two equivalent matrices have same rank.

15 Rank of The Products:

Theorem 13: The rank of the product of two matrices cannot exceed the rank of either of them. i.e.

$$\rho(AB) \leq \rho(A) \quad \text{and } \rho(AB) \leq \rho(B)$$

Proof: Assume it without proof.

Example 25: Show that $rank\ (AA') = rank\ (A)$. (Provided that A is nonsingular)

Solution: We know that

$$rank\ (A) = rank\ (A')$$

and

$$rank\ (AB) \leq rank\ B$$

Let $B = AA'$, then

$$rank\ (B) = rank\ (AA') \leq rank\ (A)$$

Thus $rank\ (AA') \leq rank\ (A)$.

Now

$$A^{-1}B = A^{-1}(AA') = A'$$

Then

$$\begin{aligned} rank\ (A) &= rank\ (A') \\ &= rank\ (A^{-1}B) \leq rank\ (B) \end{aligned}$$

Thus $rank\ (A) \leq rank\ (B)$.

Hence $rank\ (AA') = rank\ (A)$.

Remarks: Similarly we can show that $rank\ (AA^{\theta}) = rank\ (A)$.

Example 26: If A is a matrix of order $m \times n$ and R ia a non-singular matrix of order m then shoe that

$$\rho(RA) = \rho(A)$$

Solution: Let N_r be the normal form of A, then we have

$$RAC = N_r$$
$$\Rightarrow A = R^{-1}N_rC^{-1} \quad \cdots (1)$$

Since R and C are non-singular matrices hence , their inverses are possible. So let $R^{-1} = E$ and $C^{-1} = F$, which are also non-singular matrices. Hence equation (1) becomes

$$A = EN_rF$$
$$\Rightarrow RA = REN_rF$$
$$= IN_rF \quad (\because E = R^{-1}\text{and}RR^{-1} = I, \text{where I is a non-singular matrix to appropriate ord}$$
$$\Rightarrow RA \sim N_r$$
$$\therefore \rho(RA) = \rho(N_r) \quad (\because \text{Rank of equivalent matrices are same})$$
$$= \rho(A) \quad (\because \text{Rank of a matrix is equal to the rank of its normal form})$$

16 Cayley Hamilton Theorem:

Characteristic Matrix:

Let A be a square matrix. Then the matrix polynomial $A - xI$ of the first degree is called the Characteristic Matrix of A.

Characteristic Polynomial:

Let A be a square matrix. Then the determinant $|A - xI|$ which is an ordinary polynomial in x, with scalar coefficients, is called the Characteristic Polynomial of the matrix A.

e.g.

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$$

$$\Rightarrow A - xI = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - x & 3 \\ -1 & 4 - x \end{bmatrix}$$

This matrix is the characteristic matrix of A.

Also

$$|A - xI| = \begin{vmatrix} 2-x & 3 \\ -1 & 4-x \end{vmatrix}$$

$$= (2-x)(4-x) + 3$$

$$= x^2 - 6x + 11$$

Thus $|A - xI| = x^2 - 6x + 11$ is the characteristic polynomial of A.

NOTE:

1. The constant term of the characteristic polynomial $|A - xI|$ of A is $|A|$.
2. Degree of the characteristic polynomial of the matrix of order $n \times n$ is n.

Characteristic Equation:

For any square matrix A , the equation $|A - xI| = 0$ is said to be Characteristic Equation of the matrix A.

e.g. For $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ characteristic equation is $|A - xI| = x^2 - 6x + 11 = 0$

i.e. $x^2 - 6x + 11 = 0$

Theorem: State and Prove Cayley-Hamilton Theorem.

Statement: Every square matrix satisfies its characteristic equation.

OR

If A is a square matrix of order $n \times n$ and $|A - xI| = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ is the characteristic equation of A , then

$$a_0I + a_1A + a_2A^2 + \cdots + a_nA^n = 0 = O_{n\times n}$$

Proof: Let A be any square matrix of order $n \times n$.

Suppose $|A - xI| = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

Then $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ ....(1) is the characteristic equation of A.

Also suppose that

$adj.(A - xI) = B_0 + B_1x + B_2x^2 + \cdots + B_{n-1}x^{n-1}$..........(2)

Now we know that ,

For any square matrix A

$$A(adj.A) = |A| \cdot I = (adj.A)A$$

Then we have, $(A - xI)(adj.A - xI) = |A - xI|I$

$\Rightarrow (A - xI)(B_0 + B_1x + B_2x^2 + \cdots + B_{n-1}x^{n-1}) = (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)I$

($\because$ by (1) and (2))

$$\Rightarrow (AB_0 + AB_1x + AB_2x^2 + \cdots + AB_{n-1}x^{n-1}) - (B_0x + B_1x^2 + \cdots + B_{n-1}x^n)$$

$$= a_0I + a_1Ix + a_2Ix^2 + \cdots + a_nIx^n$$

$$\Rightarrow AB_0 + (AB_1 - B_0)x + (AB_2 - B_1)x^2 + (AB_3 - B_2)x^3 + \cdots + (AB_{n-1} - B_{n-2})x^{n-1} - B_{n-1}x^n$$
$$= a_0I + a_1Ix + a_2Ix^2 + \cdots + a_nIx^n$$

Comparing coefficients of equal powers of x on both sides, we get

$$\begin{aligned} AB_0 &= a_0I \\ (AB_1 - B_0) &= a_1I \\ (AB_2 - B_1) &= a_2I \\ (AB_3 - B_2) &= a_3I \\ \vdots \quad & \quad \vdots \\ (AB_{n-1} - B_{n-2}) &= a_{n-1}I \\ -B_{n-1} &= a_nI \end{aligned}$$

Pre-multiplying the 1^{st} equation with I , 2^{nd} equation with A, 3^{rd} equation with A^2 and so on and then adding we get

$$AB_0 + A^2B_1 - AB_0 + A^3B_2 - A^2B_1 + A^4B_3 - A^3B_2 + \cdots + A^nB_{n-1} - A^{n-1}B_{n-2} - A^nB_{n-1}$$
$$= a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$$

$$\Rightarrow O_{n\times n} = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$$

i.e. $a_0I + a_1A + a_2A^2 + \cdots + a_nA^n = O_{n\times n}$

Hence the theorem is proved.

Question: Find the inverse of a non-singular square matrix A using Cayley-Hamilton Theorem.

Answer:

Let A be a non-singular matrix of order $n \times n$ and
let $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ be the characteristic equation of A.

Then $a_0 = |A| \neq 0$ and by the Cayley-Hamilton Theorem we have

$$a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n = O_{n\times n} \text{ ....(1)}$$

Pre-multiplying (1) with A^{-1}, we get

$$a_0 A^{-1} I + a_1 A^{-1} A + a_2 A^{-1} A^2 + \cdots + a_n A^{-1} A^n = A^{-1} O_{n\times n}$$

$$\Rightarrow A^{-1} = \frac{-1}{a_0}(a_1 I + a_2 A + \cdots + a_n A^{n-1})$$

Example: Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify that it is satisfied by A and hence obtain A^{-1}.

Solution:

The characteristic equation of A is $|A - xI| = 0$

$$\Rightarrow \begin{vmatrix} 2-x & -1 & 1 \\ -1 & 2-x & -1 \\ 1 & -1 & 2-x \end{vmatrix} = 0$$

$$\Rightarrow (2-x)[(2-x)(2-x)-1] + 1[(-1)(2-x)+1] + 1[1-(2-x)] = 0$$

$$\Rightarrow (2-x)[4-4x+x^2-1] + (x-1) + (x-1) = 0$$

$$\Rightarrow (2-x)(x^2-4x+3) + 2x - 2 = 0$$

$$\Rightarrow 2x^2 - 8x + 6 - x^3 + 4x^2 - 3x + 2x - 2 = 0$$

$$\Rightarrow -x^3 + 6x^2 - 9x + 4 = 0 \text{.............}(1)$$

Now,

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Replacing x by A, x^2 by A^2 and x^3 by A^3 in equation (1), we get

$$-A^3 + 6A^2 - 9A + 4I$$

$$= - \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} + 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. $-A^3 + 6A^2 - 9A + 4I = O_{3\times3}$..............(2).

which shows that Cayley Hamilton theorem is verified.

By (1), $|A| =$ constant term $= 4$

i.e. $|A| \neq 0$.

$\therefore A$ is non-singular.

Multiplying equation (2) with A^{-1} , we get

$$A^{-1}(-A^3 + 6A^2 - 9A + 4I) = A^{-1}O_{3\times3}$$

$$\Rightarrow -A^2 + 6A - 9I + 4A^{-1} = O_{3\times3}$$

$$\Rightarrow A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$

$$= \frac{1}{4}(\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix})$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Example: Show that the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ satisfies Cayley-Hamilton theorem. Hence obtain A^{-1} and A^{-2}.

Solution:

The characteristic equation of A is $|A - xI| = 0$.

$$\Rightarrow \begin{vmatrix} 1-x & 2 & 0 \\ 2 & -1-x & 0 \\ 0 & 0 & -1-x \end{vmatrix} = 0$$

$$\Rightarrow (1-x)[(1+x)^2 - 0] - 2[2(-1-x) - 0] + 0 = 0$$

$$\Rightarrow (1-x)(1+2x+x^2) + 4 + 4x = 0$$

$$\Rightarrow 1 + 2x + x^2 - x - 2x^2 - x^3 + 4 + 4x = 0$$

$\Rightarrow -x^3 - x^2 + 5x + 5 = 0$

Thus $-x^3 - x^2 + 5x + 5 = 0$ is the characteristic equation of A.

Now,

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Then } A^3 = A^2 \cdot A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore -A^3 - A^2 + 5A + 5I$

$$= -\begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore -A^3 - A^2 + 5A + 5I = O_{3\times 3}$............(1)

which shows that Cayley Hamilton theorem has been verified.

Now, $|A| = 5 \neq 0$

$\therefore A$ is non-singular.

Multiplying (1) with A^{-1}, we get

$$-A^2 - A + 5I + 5A^{-1} = O_{3\times 3}$$

$$\Rightarrow 5A^{-1} = A^2 + A - 5I$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Also,

$$A^{-2} = A^{-1} \cdot A^{-1}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

$$\therefore A^{-2} = \frac{1}{25}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

Example: Verify the Calyey Hamilton theorem for the following matrix. Hence find its inverse if possible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution: The characteristic equation of A is $|A - xI| = 0$.

$$\Rightarrow \begin{vmatrix} 1-x & 2 & 3 \\ 2 & -1-x & 4 \\ 3 & 1 & 1-x \end{vmatrix} = 0$$

$$\Rightarrow (1-x)[(-1-x)(1-x)-4]-2[2(1-x)-12]+3[2-3(-1-x)]=0$$

$$\Rightarrow (1-x)[-(1-x^2)-4]-2[2-2x-12]+3[2+3+3x]=0$$

$$\Rightarrow (1-x)(x^2-5)+20+4x+15+9x=0$$

$$\Rightarrow x^2-5-x^3+5x+20+4x+15+9x=0$$

$$\Rightarrow -x^3+x^2+18x+30=0$$

now $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix}$

Also $A^3 = A^2 \cdot A = \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix}$

$\therefore -A^3 + A^2 + 18A + 30I$

$$= -\begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} + 18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} + 30\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e $-A^3 + A^2 + 18A + 30I = O_{3\times3}$.......(1)

Which shows that Cayley Hamilton theorem is verified.

Here $|A| = 18 \neq 0$, So A^{-1} exists.

Multiplying (1) by A^{-1}, we get

$$-A^2 + A + 18I + 30A^{-1} = O_{3\times3}$$

$$\Rightarrow 30A^{-1} = A^2 - A - 18I$$

$$= \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} - 18 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 1 & 11 \\ 10 & -5 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{30} \begin{bmatrix} -5 & 1 & 11 \\ 10 & -5 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

HOMEWORK: Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & -3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Hence find the inverse , if possible.

Theorem: The characteristic equation of a 2×2 matrix A is given by $\lambda^2 - S_1\lambda + |A| = 0$ where $S_1 = trace(A)$.

Proof:

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

The characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

$$\Rightarrow a_{11}a_{22} - \lambda(a_{11} + a_{22}) + \lambda^2 - a_{12}a_{21} = 0$$

$$\Rightarrow \lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\Rightarrow \lambda^2 - s_1\lambda + |A| = 0$$

Theorem: The characteristic equation of a 3×3 matrix A is given by $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$. Where $S_1 = trace(A)$, $S_2 =$ sum of the minors of the principal diagonal $= A_{11} + A_{22} + A_{33}$.

Example: Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and use it to find a simplified form of $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I_3$.

Solution:

The characteristic equation is: $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

Where $S_1 = trace(A) = 2 + 1 + 2 = 5$,

$S_2 =$ sum of the minors of the principal diagonal $= A_{11} + A_{22} + A_{33}$

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2 + 3 + 2 = 7$$

And, $|A| = 2(2) - 1(0) + 1(-1) = 3$

$\therefore$ The characteristic equation becomes $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$.

Now we show that $A^3 - 5A + 7A - 3 = O_{3\times3}$

$$A^2 = A \cdot A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

And

$$A^3 = A^2 \cdot A = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$\therefore A^3 - 5A^2 + 7A - 3I_3 = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5 \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= O_3$$

Which shows that Cayley Hamilton Theorem is verified.

Now we divide the polynomial $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I_3$ by the characteristic polynomial $A^3 - 5A^2 + 7A - 3I_3$.

Thus, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I_3$

$$= (A^5 + A)(A^3 - 5A^2 + 7A - 3I_3) + (A^2 + A + I_3)$$

$$= (A^5 + A)(O_3) + (A^2 + A + I_3)$$

$$= (A^2 + A + I_3)$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

17 System of Linear Homogeneous Equations

Consider a system of m linear equations in n unknowns which is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

such that a_{ij} is a member of a number field F for every admissible value of the suffixes i and j.This system of equations is known as the system of linear homogeneous equations.

We write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix} \quad O = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ \cdots \\ 0 \end{bmatrix}$$

so that A,X and O are $m \times n$,$n \times 1$ and $m \times 1$ matrices respectively over F. The given system of equations is thus equivalent to the single matrix equation

$$AX = O \quad ....(1)$$

The number $X = O$ is obviously a trivial solution of the matrix equation (1).

Result 1:

The equation $AX = O$ has a non-zero solution if and only if the rank r of A is less than the number n, i.e. of the unknowns.

Result 2:

A homogeneous system of linear equations necessarily possesses a non-zero solution if the number n of unknowns exceeds the number m of equations.

Result 3:

Every singular matrix is a Right as well as Left zero divisor.

The rank r of an n-rowed singular matrix being less than n, then by Result 1, there exists a non-zero matrix X such that

$$AX = O$$

Again the rank of the transpose A' of A being the same as that of A, there exists a non zero matrix Y such that

$$A'Y = O \Rightarrow Y'A = O$$

so that A is also a left zero divisor.

18 System of Linear Non-Homogeneous Equations:

Consider a system of m linear equations in n unknowns which is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

We can write this system as a single matrix equation

$$AX = B$$

where B is the column matrix with m components $b_1, b_2, \cdots, b_m$ in which atleast one b_i is non-zero. Then this system is known as system of linear non-homogeneous equations.

Defination: The system of linear equations is said to be *consistent* if it has at least one solution(either unique or infinitely many).

Defination: The system of equations is said to be *inconsistent* if it has no solution.

Condition for consistency of non-homogeneous linear equations:

Theorem: The equation

$$AX = B$$

is consistent i.e. possesses a solution if and only if the two matrices A and [A B] are of the same rank.

Corollary 1: Let A be an n-rowed non-singular square matrix.

In this case the rank of each A and [A B] is n.

Thus the equation $AX = B$ is consistent. Also the solution in this case is unique.

We already know that $A^{-1}B$ is the unique solution in this case.

Defination:

The matrix of the form

$$\tilde{A} = [A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called an augmented matrix.

Solution of the system of equations by Gauss Elimination Method:

Consider a system of linear equations $AX = B$.

Step 1 : Write the augmented matrix of the given system.

Step 2 : Obtain the Triangular form (Row Echelon Form) of the augmented matrix. Then we have three possibilities.

- If the last row is a zero row (i.e all entries on left side of augmented matrix and entry on the right side of augmented matrix is zero) then system has infinitely many solutions.
- If the entries of the last row in the left side of augmented matrix are all zero but the entry on the right side of augmented matrix is non zero then the system has no solution.

- If the augmented matrix does not have any zero row then inverse of A exists. In that case system has a unique solution which can be obtained by using back substitution.

Example: Solve the following system of equations

$$2x + y + z = 0$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

Solution: The corresponding augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$R_1 \to \dfrac{1}{2}R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$R_2 \to R_2 + (-3)R_1, R_3 \to R_3 + (-1)R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & 18 \\ 0 & \frac{7}{2} & \frac{17}{2} & 16 \end{array}\right]$$

$R_2 \to (2)R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 3 & 36 \\ 0 & \frac{7}{2} & \frac{17}{2} & 16 \end{array}\right]$$

$R_3 \to R_3 + (\frac{-7}{2})R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 3 & 36 \\ 0 & 0 & -2 & -110 \end{array}\right]$$

$R_3 \to (\frac{-1}{2})R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 3 & 36 \\ 0 & 0 & 1 & 55 \end{array}\right]$$

The resulting system is an upper triangular, which yields

$$x + \frac{1}{2}y + \frac{1}{2}z = 0$$
$$y + 3z = 36$$
$$z = 55$$

Back substitution gives the solution : $z = 55, y = -129, x = 37$.

Example : Solve the following system of equations using Gauss Elimination method.

$$x - 2y + w = 3$$
$$-x + 2y + z - \frac{1}{2}w = -7$$
$$4x - 8y + 6z + 7w = -3$$

Solution: The corresponding augmented matrix is given by

$$[A|B] = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 3 \\ -1 & 2 & 1 & \frac{-1}{2} & -7 \\ 4 & -8 & 6 & 7 & -3 \end{array}\right]$$

$R_2 \to R_2 + R_1, R_3 \to R_3 + (-4)R_1$

$$[A|B] = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & \frac{1}{2} & -4 \\ 0 & 0 & 6 & 3 & -15 \end{array}\right]$$

$R_3 \to R_3 + (-6)R_2$

$$[A|B] = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & \frac{1}{2} & -4 \\ 0 & 0 & 0 & 0 & 9 \end{array}\right]$$

The resultant system is

$$\begin{aligned} x - 2y + w &= 3 \\ z + \frac{1}{2}w &= -4 \\ 0 &= 9 \end{aligned}$$

Which is not possible. Thus the system is inconsistent.

Example: Consider the following system:

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}$$

For what values of λ and μ do the system has (i) no solution (ii) unique solution (iii) infinite solution.

Solution: The corresponding augmented matrix is given by

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R - 1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$$

(i) If $\lambda - 3 = 0$ and $\mu - 10 \neq 0$, i.e. if $\lambda = 3$ and $\mu \neq 10$ then the system does not have any solution.

(ii) If $\lambda - 3 \neq 0$ then the system has a unique solution. That is $\lambda \neq 3$ and μ can possess any real value.

(iii) If $\lambda - 3 = 0$ and $\mu - 10 = 0$ that is $\lambda = 3$ and $\mu = 10$ then the system has infinite solution.

Example: Solve the following system for x, y and z:

$$-\frac{1}{x}+\frac{3}{y}+\frac{4}{z}=30, \frac{3}{x}+\frac{2}{y}-\frac{1}{z}=9, \frac{2}{x}-\frac{1}{y}+\frac{2}{z}=10$$

Solution: Let $\frac{1}{x}=x', \frac{1}{y}=y', \frac{1}{z}=z'$.

Thus we have

$$-x'+3y'+4z'=30, 3x'+2y'-z'=9, 2x'-y'+2z'=10$$

Using Gauss Elimination method we get,

$$[A|B] = \left[\begin{array}{ccc|c} -1 & 3 & 4 & 30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right] \quad R_1 \to (-1)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right] \quad R_2 \to R_2 + (-3)R_1, R_3 \to R_3 + (-2)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 11 & 11 & 99 \\ 0 & 5 & 10 & 70 \end{array}\right] \quad R_2 \to \frac{1}{11}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 5 & 10 & 70 \end{array}\right] \quad R_3 \to R_3 + (-5)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 5 & 25 \end{array}\right] \quad R_3 \to \frac{1}{5}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The resultant system is

$$x' - 3y' - 4z' = -30$$
$$y' + z' = 9$$
$$z' = 5$$

By backward substitution we get ,

$z' = 5 \Rightarrow y' = 9 - 5 = 4 \Rightarrow x' = -30 + 3(4) + 4(5) = 2$

$\therefore x = \frac{1}{2}, y = \frac{1}{4}, z = \frac{1}{5}$

Example: What conditions must b_1, b_2, b_3 satisfy in order for,

$$x_1 + 2x_2 + 3x - 3 = b_1, 2x_1 + 5x_2 + 3x_3 = b_2, x_1 + 8x_3 = b - 3$$

be consistent ?

Solution: Solving the system by Gauss Elimination Method we get,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{array}\right] \quad R_2 \to R_2 + (-2)R_1, R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & -3 & b_2 - 2b_1 \\ 0 & -2 & 5 & b_3 - b_1 \end{array}\right] R_3 \to R_3 + 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & -3 & b_2 - 2b_1 \\ 0 & 0 & -1 & b_3 + 2b_2 - 5b_1 \end{array}\right] R_3 \to (-1)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & -3 & b_2 - 2b_1 \\ 0 & 0 & 1 & -b_3 - 2b_2 + 5b_1 \end{array}\right]$$

Since inverse of A exists, the solution is unique for every choice of the values of b_1, b_2, b_3.

Example: Solve the following system by Gauss Elimination method:

$$2x + 2y + 2z = 0, -2x + 5y + 2z = 1, 8x + y + 4z = -1$$

Solution: The corresponding augmented matrix is given by,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right] \quad R_1 \to \frac{1}{2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right] \quad R_2 \to R_2 + 2R - 1, R_3 \to R_3 + (-8)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right] \quad R_2 \to \frac{1}{7}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & -7 & -4 & -1 \end{array}\right] \quad R_3 \to R_3 + 7R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Note that the last row is zero row , so the system has infinitely many solutions. The resultant system is given by,

$$x + y + z = 0, y + \frac{4}{7}z = \frac{1}{7}$$

Let $z = k \in \mathbb{R}$.

$\therefore y = \frac{1}{7} - \frac{4}{7}k$ and $x = -\frac{1}{7} + \frac{4}{7}k - k = -\frac{1}{7} - \frac{3}{7}k$

$\therefore$ The solution set is $\{(-\frac{1}{7} - \frac{3}{7}k, \frac{1}{7} - \frac{4}{7}k, k) : k \in \mathbb{R}\}$.

19 Solving Homogeneous Linear System of Equations:

Example: Solve the following system of equations:

$$4x + 3y - z = 0, 3x + 4y + z = 0, 5x + y - 4z = 0$$

Solution: The corresponding augmented matrix is given by,

$$[A|O] = \left[\begin{array}{ccc|c} 4 & 3 & -1 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & -4 & 0 \end{array}\right] \quad R_1 \to R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & -4 & 0 \end{array}\right] \quad R_2 \to R_2 + (-3)R_1, R_3 \to R_3 + (-5)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 6 & 6 & 0 \end{array}\right] \quad R_2 \to \frac{1}{7}R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 6 & 6 & 0 \end{array}\right] \quad R_1 \to R_1 + R_2, R_3 \to R_3 + (-6)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last row is zero row. (i.e. $rank(A) < 3$). Hence the system has a non trivial solution.

The resultant system is

$$x - z = 0$$

$$y + z = 0$$

Let $z = k \Rightarrow y = -k, x = k$.

$\therefore$ The solution set is $\{(k, -k, k) | k \in \mathbb{R}\}$.

Example: Find the value of λ so that the equations:

$$2x + y + 2z = 0$$

$$x + y + 3z = 0$$

$$4x + 3y + \lambda z = 0$$

have a non-trivial solution.

Solution: The corresponding augmented matrix is given by

$$[A|O] = \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & \lambda & 0 \end{array}\right] \quad R_2 \to R_2 + (-2)R_1, R_3 \to R_3 + (-4)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & \lambda - 12 & 0 \end{array}\right] \quad R_2 \to (-1)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -1 & \lambda - 12 & 0 \end{array}\right] \quad R_3 \to R_3 + R_2, R_1 \to R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & \lambda - 8 & 0 \end{array}\right]$$

The system will have a non-trivial solution if the last row is zero row.

Then we must have $\lambda - 8 = 0 \Rightarrow \lambda = 8$.

20 Characteristic Roots And Characteristic Vectors of a Matrix

Characteristic Roots and Characteristic Vectors of a Square Matrix:

Defination: Any non-zero vector, X is said to be a Characteristic Vector of a matrix A is there exists a number λ such that

$$AX = \lambda X$$

Also then λ is called the Characteristic Root of the matrix A corresponding to the characteristic vector X and vice versa.

NOTE: Characteristic Roots(Vectors) are also called *Proper, Latent* or *Eigen values(vectors).*

Theorem: A characteristic vector of a matrix cannot correspond to two different characteristic roots.

Proof: Let X be a characteristic vector of A corresponding to two different characteristic roots λ_1 and λ_2.

Then $AX = \lambda_1 X$ and $AX = \lambda_2 X$.

$\Rightarrow \lambda_1 X = \lambda_2 x$

$\Rightarrow (\lambda_1 - \lambda_2)X = O$

$\Rightarrow X = O$ or $\lambda_1 - \lambda_2 = 0$

But $X \neq O$.

Thus $\lambda_1 - \lambda_2 = 0$.

i.e $\lambda_1 = \lambda_2$.

Remark: From above result , a characteristic vector of a matrix corresponds to a unique characteristic root, but a characteristic root can correspond to different characteristic vectors.

If λ is a characteristic root of matrix A corresponding to characteristic vector X,

then $AX = \lambda X$.

For $k \neq 0$ $A(kX) = k(AX) = k(\lambda X) = \lambda(kX)$.

$\therefore$ kX is also a characteristic vector corresponding to λ.

Defination: The set of all characteristic roots of the matrix A is called the spectrum of A.

Defination: Let A be any $n \times n$ matrix and λ be any eigen value for A. Then the set $E_\lambda = \{X | AX = \lambda X\}$ is called the eigen space of λ.

Determination of Characteristic Roots and Vectors:

Theorem: Let A be an $n \times n$ matrix and λ be a real number. Then λ is an eigen value of A if and only if $|A - \lambda I| = 0$.Thus the eigen value is a solution of the characteristic equation of A. The eigen vectors corresponding to λ are the non-trivial (i.e. non-zero) solutions of the system $(A - \lambda I_{n \times n})X = 0$

Remark: For the non-trivial solution of the system $(A - \lambda I)X = O$ the rank of the matrix $A - \lambda I$ should be less than n. So for a 3×3 matrix A we may solve any two (non-parallel) equations using Cramer's Rule to find the eigen vectors.

Example: Find Eigen Values and Eigen Vectors of

$$A = \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}$$

Solution:

The characteristic equation is $|A - \lambda I_3| = 0$

$$\therefore \begin{vmatrix} -4-\lambda & 8 & -12 \\ 6 & -6-\lambda & 12 \\ 6 & -8 & 14-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-4-\lambda)[(-6-\lambda)(14-\lambda)+96]-8[6(14-\lambda)-72]-12[-48-6(-6-\lambda)] = 0$$

$$\Rightarrow (-4-\lambda)[-84+6\lambda-14\lambda+\lambda^2+96]-8[84-6\lambda-72]-12[-48+36+6\lambda] = 0$$

$$\Rightarrow (-4-\lambda)(\lambda^2-8\lambda+12)-8(-6\lambda+12)-12(6\lambda-12) = 0$$

$$\Rightarrow -4\lambda^2+32\lambda-48-\lambda^3+8\lambda^2-12\lambda+48\lambda-96-72\lambda+144 = 0$$

$$\Rightarrow -\lambda^3+4\lambda^2-4\lambda = 0$$

$$\Rightarrow \lambda(\lambda-2)^2 = 0$$

$\therefore$ The eigen values of the matrix A is given by $\lambda = 0, 2$.

To find Eigen Vector corresponding to $\lambda = 0$:

We solve the system $(A - 0I)X = O$

$$\Rightarrow \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

We have

$$-4x + 8y - 12z = 0$$
$$6x - 6y + 12z = 0$$
$$6x - 8y + 14z = 0$$

By Cramer's Rule we get,

$$\frac{x}{\begin{vmatrix} 8 & -12 \\ -6 & 12 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -4 & -12 \\ 6 & 12 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -4 & 8 \\ 6 & -6 \end{vmatrix}} = k \in \mathbb{R}$$

$$\therefore \frac{x}{24} = \frac{-y}{24} = \frac{z}{-24}$$

$$\therefore \frac{x}{1} = \frac{y}{-1} = \frac{z}{-1} = k$$

$\therefore$ The eigen vectors corresponding to $\lambda = 0$ is given by

$$X = k \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} ; k \in \mathbb{R} - \{0\}$$

$\therefore$ The eigen space for $\lambda_1 = 0$ is

$E_0 = \{\alpha(1, -1, -1)/\alpha \in \mathbb{R}\}$

To find eigen vectors corresponding to $\lambda = 2$:

We solve the system $(A - 2I)X = O$

$$\Rightarrow \begin{bmatrix} -6 & 8 & -12 \\ 6 & -8 & 12 \\ 6 & -8 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

We have

$$-6x + 8y - 12z = 0$$

$$6x - 8y + 12z = 0$$

$$6x - 8y_2 + 12z = 0$$

All three equations are same. So we solve any one of them.

Let us consider

$$-6x + 8y - 12z = 0$$

$$\therefore x = \frac{4}{3}y - 2z$$

$\therefore$ Then eigen vectors corresponding to $\lambda = 2$ is given by ,

$$X = \begin{bmatrix} \frac{4}{3}y - 2z \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3}y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -2z \\ 0 \\ z \end{bmatrix}$$

$$= y \begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

where $y, z \in \mathbb{R} - \{0\}$

The eigen space is given by $E_2 = \{y(\frac{4}{3}, 1, 0) + z(-2, 0, 1) : y, z \in \mathbb{R}\}$

Example: Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: The characteristic equation is $|A - \lambda I_3| = 0$

$$\therefore \begin{vmatrix} -2-\lambda & -8 & -12 \\ 1 & 4-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (-2-\lambda)[(4-\lambda)(1-\lambda) - 0] + 8[1(1-\lambda) - 0] - 12(0) = 0$

$\Rightarrow (-2-\lambda)[4 - 5\lambda + \lambda^2] + 8 - 8\lambda = 0$

$\Rightarrow -8 + 10\lambda - 2\lambda^2 - 4\lambda + 5\lambda^2 - \lambda^3 + 8 - 8\lambda = 0$

$\Rightarrow -\lambda^3 + 3\lambda^2 - 2\lambda = 0$

$\Rightarrow (-\lambda)(\lambda - 2)(\lambda - 1) = 0.$

$\therefore$ The eigen values of A are

$$\lambda = 0, 1, 2$$

To find eigen vectors corresponding to $\lambda = 0$:

We solve the system $(A - 0I) = O$

$$\Rightarrow \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O.$$

This results into the following system of equations:

$$\begin{aligned} -2x - 8y - 12z &= 0 \\ x + 4y + 4z &= 0 \\ z &= 0 \end{aligned}$$

We solve the first two equations using Cramer's Rule:

$$\frac{x}{\begin{vmatrix} -8 & -12 \\ 4 & 4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & -12 \\ 1 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -8 \\ 1 & 4 \end{vmatrix}}$$

$$\therefore \frac{x}{16} = \frac{-y}{4} = \frac{z}{0}$$

$$\therefore \frac{x}{4} = \frac{-y}{1} = \frac{z}{0}$$

$\therefore$ The eigen vector corresponding to $\lambda = 0$ is given by

$$X = k \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \text{ where } k \in \mathbb{R} - \{0\}.$$

To find eigen vectors corresponding to $\lambda = 1$:

We solve the system $(A - 1I)X = O$

$$\Rightarrow \begin{bmatrix} -3 & -8 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results into the following system of equations:

$$\begin{aligned} -3x - 8y - 12z &= 0 \\ x + 3y + 4z &= 0 \\ 0 &= 0 \end{aligned}$$

We solve first two equations using Cramer's Rule,

$$\frac{x}{\begin{vmatrix} -8 & -12 \\ 3 & 4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -3 & -12 \\ 1 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & -8 \\ 1 & 3 \end{vmatrix}}$$

$$\therefore \frac{x}{4} = \frac{-y}{0} = \frac{z}{-1}$$

$\therefore$ The eigen vector corresponding to $\lambda = 1$ is given by,

$$X = k \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} \quad k \in \mathbb{R} - \{0\}$$

To find eigen vectors corresponding to $\lambda = 2$:

We solve the system $(A - 2I)X = O$

$$\Rightarrow \begin{bmatrix} -4 & -8 & -12 \\ 1 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This results into the following system of equations:

$$-4x - 8y - 12z = 0$$
$$x + 2y + 4z = 0$$
$$-z = 0$$

We solve first two equations using Cramer's Rule,

$$\frac{x}{\begin{vmatrix} -8 & -12 \\ 2 & 4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -4 & -12 \\ 1 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -4 & -8 \\ 1 & 2 \end{vmatrix}}$$

$$\therefore \frac{x}{-8} = \frac{-y}{-4} = \frac{z}{0}$$

$$\therefore \frac{x}{-2} = \frac{y}{1} = \frac{z}{0}$$

$\therefore$ The eigen vector corresponding to $\lambda = 1$ is given by,

$$X = k \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad k \in \mathbb{R} - \{0\}$$

Example: Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Solution:

The characteristic equation is $|A - \lambda I| = 0$

Which is $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

where

$S_1 = trace(A) = 11$

$S_2 =$ Sum of minors of diagonal element $= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix} =$ $14 + 8 + 14 = 36$

$|A| = 3(14) + 1(-2) + (-4) = 42 - 6 = 36$

$\therefore$ The characteristic equation is given by

$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$\Rightarrow (\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$

$\therefore \quad \lambda = 2, 3, 6$ are the eigen values of A.

To find eigen vectors corresponding to $\lambda = 2$:

We solve the system $(A - 2I) = O$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O.$$

This results into the following system of equations:

$$\begin{aligned} x - y + z &= 0 \\ -x + 3y - z &= 0 \\ x - y + z &= 0 \end{aligned}$$

We solve the first two equations using Cramer's Rule:

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$$

$$\therefore \frac{x}{-2} = \frac{-y}{0} = \frac{z}{2}$$

$$\therefore \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

$\therefore$ The eigen vector corresponding to $\lambda = 2$ is given by

$$X = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \text{ where } k \in \mathbb{R} - \{0\}.$$

To find eigen vectors corresponding to $\lambda = 3$:

We solve the system $(A - 3I) = O$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O.$$

This results into the following system of equations:

$$0x - y + z = 0$$
$$-x + 2y - z = 0$$
$$x - y + 0z = 0$$

We solve the first two equations using Cramer's Rule:

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\therefore \frac{x}{-1} = \frac{-y}{1} = \frac{z}{-1}$$

$$\therefore \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$\therefore$ The eigen vector corresponding to $\lambda = 3$ is given by

$$X = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ where } k \in \mathbb{R} - \{0\}.$$

To find eigen vectors corresponding to $\lambda = 6$:

We solve the system $(A - 6I) = O$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O.$$

This results into the following system of equations:

$$-3x - y + z = 0$$
$$-x - y - z = 0$$
$$x - y - 3z = 0$$

We solve the first two equations using Cramer's Rule:

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\therefore \frac{x}{2} = \frac{-y}{4} = \frac{z}{2}$$
$$\therefore \frac{x}{1} = \frac{y}{-2} = \frac{z}{1}$$

$\therefore$ The eigen vector corresponding to $\lambda = 6$ is given by

$$X = k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \text{ where } k \in \mathbb{R} - \{0\}.$$

Example: Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution:

The characteristic equation is $|A - \lambda I| = 0$

Which is $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

where

$S_1 = trace(A) = -1$

$S_2 =$ Sum of minors of diagonal element $= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} =$
$-12 - 3 - 6 = -21$

$|A| = -2(-12) - 2(-6) - 3(-3) = 24 + 12 + 9 = 45$

$\therefore$ The characteristic equation is given by

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\Rightarrow (\lambda + 3)^2(\lambda - 5) = 0$

$\therefore$ The eigen values of A is given by $\lambda = -3, 5$

To find eigen vectors corresponding to $\lambda = 5$:

We solve the system $(A - 5I) = O$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O.$$

This results into the following system of equations:

$$-7x + 2y - 3z = 0$$
$$2x - 4y - 6z = 0$$
$$-x - 2y - 5z = 0$$

We solve the first two equations using Cramer's Rule:

$$\frac{x}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\therefore \frac{x}{-24} = \frac{-y}{48} = \frac{z}{24}$$

$$\therefore \frac{x}{-1} = \frac{y}{-2} = \frac{z}{1}$$

$\therefore$ The eigen vector corresponding to $\lambda = 5$ is given by

$$X = k \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \text{ where } k \in \mathbb{R} - \{0\}.$$

To find eigen vector corresponding to $\lambda = -3$:

We solve the system $(A + 3I)X = O$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

This result into the following system of equations:

$$x + 2y - 3z = 0$$
$$2x + 4y - 6z = 0$$
$$-x - 2y + 3z = 0$$

All three equations are same so we solve any one of them.

Let us consider

$$x + 2y - 3z = 0$$

$$\therefore x = -2y + 3z$$

$\therefore$ The eigen vector corresponding to $\lambda = -3$ is given by,

$$X = \begin{bmatrix} -2y + 3z \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} -2y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 3z \\ 0 \\ z \end{bmatrix}$$

$$= y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

where $y, z \in \mathbb{R} - \{0\}$.

HOMEWORK: Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

21 Nature of the Characteristic Roots of Some Special Types of Matrices:

Theorem: The characteristic roots of a Hermitian matrices are all real.

Proof: let λ be a characteristic root of a hermitian matrix A. Then there is a non-zero vector X such that

$$AX = \lambda X$$

Pre multiplying with X^θ we get,

$$X^\theta AX = X^\theta \lambda X = \lambda X^\theta IX$$

$$\text{i.e. } X^\theta AX = \lambda X^\theta IX \text{............}(1)$$

Now $X^\theta AX$ and $X^\theta IX$ are real numbers.

Also $X^\theta X \neq 0$ because $X \neq \overline{O}$.

$\therefore$ By (1) $\lambda = \dfrac{X^\theta AX}{X^\theta IX}$ is real.

Corollary : The characteristic roots of a real symmetric matrix are all real.

Proof: Let A be any real symmetric matrix.

Then

$$\begin{aligned} A^\theta = (\overline{A})' = A' \quad & (\overline{A} = A \text{ as A is real}) \\ = A \quad & (\because \text{A is symmetric}) \end{aligned}$$

i.e. A is a Hermitian Matrix.

$\therefore$ By above theorem , the characteristic roots of A are all real.

Corollary : A characteristic root of a Skew-Hermitian matrix is either zero or a purely imaginary number.

Proof: Let A be any Skew-Hermitian matrix and λ be a characteristic root of A.

Then there is a non-zero vector X such that

$$AX = \lambda X$$
$$\Rightarrow i(AX) = i(\lambda X)$$
$$\Rightarrow (iA)X = (i\lambda)X$$

Now, $(iA)^\theta = \bar{i}A^\theta = -i(-A) \quad (\because \bar{i} = -i \ A^\theta = -A)$

i.e. $(iA)^\theta = iA$.

$\therefore$ iA is a Hermitian and $i\lambda$ is a characteristic root of iA.

$\therefore$ By above theorem $i\lambda$ is real.

$\Rightarrow \lambda = 0$ or $\lambda =$purely imaginary number.

Corollary: A characteristic root of a real skew-symmetric matrix is either zero or a purely imaginary number.

Proof: Let A be a real skew-symmetric matrix.

Then

$$A^{\theta} = (\overline{A})'$$
$$= A' \quad (\because \overline{A} = A \quad \text{as A is real})$$
$$= -A \quad (\because A \text{is skew-symmetric})$$
$$\text{i.e.} A^{\theta} = -A$$

$\therefore$ A is skew-Hermitian matrix.

$\therefore$ By above corollary the characteristic root of A is either zero or purely imaginary number.

Theorem: The modulus of characteristic root of a unitary matrix is unity. i.e. 1.

Proof: Let A be any unitary matrix.

$$\therefore A^{\theta} A = I$$

Also let λ be a characteristic root of A and X be the corresponding characteristic vector.

Then

$$(AX) = \lambda A$$

Taking the conjugate transpose on both sides, we get

$$(AX)^\theta = (\lambda X)^\theta$$

$$\Rightarrow X^\theta A^\theta = \overline{\lambda} X^\theta$$

$$\Rightarrow X^\theta A^\theta AX = \overline{\lambda} X^\theta \lambda X$$

$$\Rightarrow X^\theta IX = \overline{\lambda}\lambda X^\theta X \quad (\because A^\theta A = I)$$

$$\Rightarrow X^\theta X = \overline{\lambda}\lambda X^\theta X$$

$$\Rightarrow (1 - \overline{\lambda}\lambda) X^\theta X = 0.................(1)$$

Now, $X \neq \overline{O} \Rightarrow X^\theta X \neq 0$

$\therefore$ By (1) we must have

$$1 - \overline{\lambda}\lambda = 0$$

$$\Rightarrow \overline{\lambda}\lambda = 1$$

$$\Rightarrow |\lambda|^2 = 1$$

$$\text{i.e.} |\lambda| = 1$$

Corollary: The modulus of each characteristic root of an orthogonal matrix is unity.

Proof: Let A be an orthogonal matrix.

Then $A'A = I$ and A is real.

i.e. $A'A = I$ and $\overline{A} = A$.

$\therefore A^{\theta}A = I.$

$\therefore$ A is a unitary matrix.

$\therefore$ By above theorem modulus of characteristic roots of A is unity.

22 The Construction of Orthogonal Matrices:

Theorem: If S is a skew symmetric matrix then $I - S$ is non-singular and the matrix

$$A = (I + S)(I - S)^{-1}$$

is orthogonal.

Solution: First we show that $I - S$ is non-singular.
If possible assume that $I - S$ is singular matrix.
Then

$$\begin{aligned} |I - S| &= 0 \\ \Rightarrow |(-1)(S - I)| &= 0 \\ \Rightarrow (-1)^n|S - I| &= 0 \\ \Rightarrow |S - I| &= 0 \end{aligned}$$

Then 1 is an eigen value of S which is not possible as S is a skew symmetric matrix , eigen values of S can be either zero or purely imaginary number. Thus,

$$|I - S| \neq 0$$

Hence $I - S$ is a non-singular matrix.

Next we show that $A = (I + S)(I - S)^{-1}$ is an orthogonal matrix.

Now,

$$A' = [(I + S)(I - S)^{-1}]' = ((I - S)^{-1})'(I + S)' = ((I - S)')^{-1}(I + S)'$$

But,

$$(I - S)' = I' - S' = I + S$$

And,

$$(I + S)' = I' + S' = I - S$$

$\therefore \quad A' = (I + S)^{-1}(I - S)$

$$\begin{aligned} \therefore A'A &= (I + S)^{-1}(I - S)(I + S)(I - S)^{-1} \\ &= (I + S)^{-1}(I + S)(I - S)(I - S)^{-1} \\ &= I \end{aligned}$$

Thus A is an orthogonal matrix.

Theorem: Every orthogonal matrix A can be expressed as

$$A = (I + S)(I - S)^{-1}$$

by a suitable choice of a real skew-symmetric matrix S provided that -1 is not a characteristic root of A.

Proof: To prove the theorem it is sufficient to show that for an orthogonal matrix A such that -1 is not a characteristic root of A, such that

$$A = (I + S)(I - S)^{-1}$$

determines a skew symmetric matrix S. Now

$$A = (I + S)(I - S)^{-1}$$

$$\Rightarrow A(I - S) = I + S$$

$$\Rightarrow A - AS = I + S$$

$$\Rightarrow A - I = (A + I)S ....(1)$$

Since -1 is not a characteristic root of A we have

$$|A - (-1)I| \neq 0$$

Therefore,

$$|A + I| \neq 0$$

Hence, $(A + I)^{-1}$ exists. Therefore, premultiplying with $(A + I)^{-1}$ on both sides of (1) we get,

$$S = (A + I)^{-1}(A - I)$$

This establishes existence of S. Finally we show that S is a real skew symmetric matrix.

$$\begin{aligned}
S' &= [(A+I)^{-1}(A-I)]' \\
&= (A-I)'[(A+I)^{-1}]' \\
&= (A-I)'[(A+I)']^{-1} \\
&= (A'-I)(A'+I)^{-1} \\
&= (A'+I)^{-1}(A'-I) \\
&= (A'+A'A)^{-1}(A'-A'A) \\
&= [A'(I+A)]^{-1}[A'(I-A)] \\
&= (I+A)^{-1}(A')^{-1}A'(I-A) \\
&= (I+A)^{-1}(I-A) \\
&= -(A+I)^{-1}(A-I) \\
&= -S
\end{aligned}$$

Hence, S is a skew symmetric matrix.

Example: If A is non-singular, prove that the eigen values of A^{-1} are the reciprocals of the eigen value of A.

Solution: Let λ be the eigen value of A and X be a corresponding eigen vector.

Then,

$$AX = \lambda X \Rightarrow X = A^{-1}(\lambda X) = \lambda(A^{-1}X)$$
$$\Rightarrow \frac{1}{\lambda}X = A^{-1}X$$
$$\Rightarrow A^{-1}X = \frac{1}{\lambda}X$$

Therefore, $\frac{1}{\lambda}$ is an eigen value of A^{-1} and X is corresponding eigen vector.

Conversely, suppose that k is an eigen value of A^{-1}. Since A is non-singular therefore A^{-1} is non-singular $(A^{-1})^{-1} = A$. Therefore it follows from the first part that $\frac{1}{k}$ is an eigen value of A. Thus each eigen value of A^{-1} is equal to the reciprocal of some eigen value of A.

Example: Show that a characteristic vector X, corresponding to a characteristic root λ of a matrix A is also a characteristic vector of every vector of every matrix $f(A)$; $f(x)$ being any scaler polynomial, and the corresponding root for $f(A)$ is f(λ). In general show that if $g(x) = \frac{f_1(x)}{f_2(x)}$; where $|f_2(A)| \neq 0$ then g(λ) is a characteristic root of $g(A) = f_1(A)f_2(A)^{-1}$.

Solution: Let λ be a characteristic root and X $\neq 0$ be corresponding characteristic vector of matrix A. Therefore,

$$AX = \lambda X$$

Now,

$$A^2X = A(AX) = A(\lambda X) = \lambda(AX) = \lambda^2 X$$

Repeating the process k times, we get,

$$A^k X = \lambda^k X$$

If $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_m x^m$ is a scaler polynomial then we have,

$$\begin{aligned}
f(A)X &= (a_0 I + a_1 A + a_2 A^2 + ... + a_m A^m)X \\
&= a_0 X + a_1 AX + a_2 A^2 X + ... + a_m A^m X \\
&= a_0 X + a_1 \lambda X + a_2 \lambda^2 X + ... + a_m \lambda^m X \\
&= (a_0 + a_1 \lambda + a_2 \lambda^2 + ... + a_m \lambda^m)X \\
\therefore f(A)X &= f(\lambda)X
\end{aligned}$$

Therefore, $f(\lambda)$ is a characteristic root of $f(A)$ corresponding to characteristic vector X. Hence, $f_1(\lambda)$ and $f_2(\lambda)$ are characteristic roots of $f_1(A)$ and $f_2(A)$ respectively. Therefore,

$$f_1(A)X = f_2(\lambda)X \text{ and } f_1(A)X = f_2(\lambda)X$$

Now, if $|f_2(A)| \neq 0$ then $f_2(A)$ is a non-singular matrix and hence characteristic roots of $f_2(A)$ are non-zero. Therefore,

$$f_2(\lambda) \neq 0$$

Hence, we also have $f_2(A)^{-1}X = f_2(\lambda)^{-1}X$ Now, if $g(A) = f_1(A)f_2(A)^{-1}$ then

$$\begin{aligned} g(A)X &= f_1(A)f_2(A)^{-1}X \\ &= f_1(A)f_2(\lambda)^{-1}X \\ &= f_2(\lambda)^{-1}(f_1(A)X) \\ &= f_2(\lambda)^{-1}(f_1(\lambda)X) \\ &= f_1(\lambda)f_2(\lambda)^{-1}X \\ &= g(\lambda)X \end{aligned}$$

Thus, $g(\lambda)$ is a characteristic root and X is corresponding characteristic vector of $g(A) = f_1(A)f_2(A)^{-1}$.

Example: Show that the two matrices A and $P^{-1}AP$ have the same characteristic roots.

Solution: Let A be a square matrix and P be a non-singular matrix of same order.
Suppose $B = P^{-1}AP$.
Now,

$$B - xI = P^{-1}AP - xI = P^{-1}AP - P^{-1}(xI)P = P^{-1}(A - xI)P$$

Therefore,

$$\begin{aligned}|B - xI| &= |P^{-1}(A - xI)P| \\ &= |P^{-1}||(A - xI)||P| \\ &= |P^{-1}||P||(A - xI)| \\ &= |P^{-1}P||(A - xI)| \\ &= |I||(A - xI)| \\ &= |(A - xI)|\end{aligned}$$

Therefore,

$$|B - xI| = 0 \Leftrightarrow |A - xI| = 0$$

Therefore, $P^{-1}AP$ and A have same characteristic equations. Hence, $P^{-1}AP$ and A have same characteristic roots.

Example: Show that the characteristics roots of A^{θ} are the conjugates of the characteristic roots of A.

Solution: Let λ be the characteristic root and $X \neq 0$ be corresponding characteristic vector of a square matrix A.
Now,

$$\begin{aligned}|A^{\theta} - \bar{\lambda}I| &= |(A - \lambda I)^{\theta}| \\ &= |\overline{(A - \lambda I)'}| \\ &= |\overline{(A - \lambda I)}|\end{aligned}$$

Therefore,

$$|A^\theta - \bar{\lambda} I| = 0 \Leftrightarrow |\overline{(A - \lambda I)}| = 0 \Leftrightarrow |A - \lambda I| = 0$$

Hence, $\bar{\lambda}$ is a characteristic root of A^θ whenever λ is a characteristic root of A.

Remark: With similar arguments we can show that if λ is an eigen value of A then λ is also an eigen value of A'.

Example: Show that the characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

Solution: Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$\therefore$

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} - \lambda \end{vmatrix}$$

$$= (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$$

Hence the characteristic roots of A are $a_{11}, a_{22}, \cdots, a_{nn}$, which are just the diagonal elements of A.

Example: Show that zero is an eigen value of a matrix A if and only if A is singular matrix.

Solution: Let A be given matrix.

Assume that 0 is an eigen value of A.

Then

$$|A - 0I| = 0$$

$$\Rightarrow |A| = 0$$

$\therefore$ A is a singular matrix.

Conversely, let A be a singular matrix $\Rightarrow |A| = 0$

$\therefore \lambda = 0$ satisfies the equation $|A - \lambda I| = 0$

$\therefore \lambda = 0$ is an eigen value of A.

Example: If α is a characteristic root of a non-singular matrix A, the prove that $\frac{|A|}{\alpha}$ is a characteristic root of *adj*. A.

Solution: Since α is a characteristic root of a non-singular matrix, therefore $\alpha \neq 0$. Also α is a characteristic root of A, then there exists a non zero vector X such that

$$AX = \alpha X$$
$$\Rightarrow (adj.\ A)AX = (adj.\ A)\alpha X$$
$$\Rightarrow [(adj.\ A)A]X = \alpha[(adj.\ A)X]$$
$$\Rightarrow |A|X = \alpha[(adj.\ A)X]$$
$$\Rightarrow \frac{|A|}{\alpha}X = (adj.\ A)X$$

Since X is a non zero vector, $\frac{|A|}{\alpha}$ is a characteristic root of $(adj.\ A)$.

HOMEWORK: Show that the characteristic roots of an idempotent matrix are either zero or unity.

NOTES:

(i) The trace of a matrix A is the sum of the eigen values of A. i.e. if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigen values of A then $tr.\ A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

(ii) The determinant of a matrix A is the product of the eigen values of A.i.e. if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigen values of A then $|A| = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$.