# Course: US01CPHY01

# UNIT – 2 ELASTICITY – II

## ❖ Introduction:

We discussed fundamental concept of properties of matter in first unit. This concept will be more use full for calculating various properties of mechanics of solid material. In this unit, we shall study the detail theory and its related experimental methods for determination of elastic constants and other related properties.

## ❖ Twisting couple on a cylinder or wire:

Consider a cylindrical rod of length l radius r and coefficient of rigidity $\eta$. Its upper end is fixed and a couple is applied in a plane perpendicular to its length at lower end as shown in fig.(a)

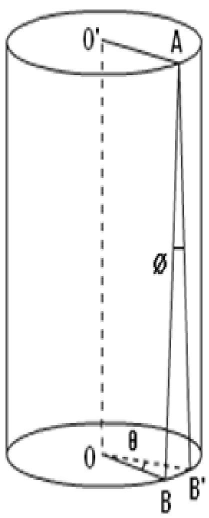



Figure : (a)

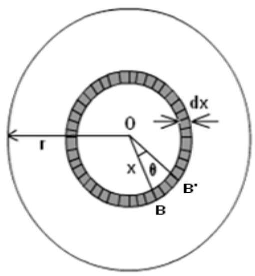



Figure : (b)

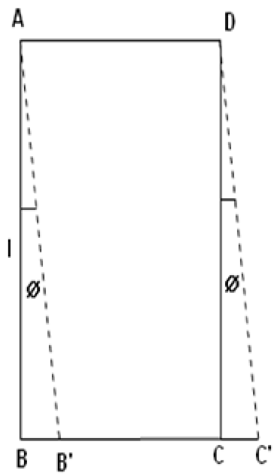



Figure : (c)

Consider a cylinder is consisting a large number of co-axial hollow cylinder. Now, consider a one hollow cylinder of radius x and radial thickness dx as shown in fig.(b). Let $\theta$ is the twisting angle. The displacement is greatest at the rim and decreases as the center is approached where it becomes zero.

As shown in fig.(a), Let AB be the line parallel to the axis OO' before twist produced and on twisted B shifts to B', then line AB become AB'.

Before twisting if hollow cylinder cut along AB and flatted out, it will form the rectangular ABCD as shown in fig.(c). But if it will be cut after twisting it takes the shape of a parallelogram AB'C'D.

The angle of shear $\angle BAB' = \emptyset$

From fig.(c) $BB' = l\emptyset$

From fig.(b) $BB' = x\theta$

$\therefore l\emptyset = x\theta$

$$\therefore\ \emptyset = \frac{x\theta}{l} \qquad \text{… … … … … … (1)}$$

The modules of rigidity is

$$\eta = \frac{Shearing\ spress}{angle\ of\ shear} = \frac{F}{\phi}$$

$$\therefore\ F = \eta \cdot \emptyset = \frac{\eta x\theta}{l}$$

The surface area of this hollow cylinder $= 2\pi x dx$

∴ Total shearing force on this area

$$= 2\pi x dx \cdot \frac{\eta x\theta}{l}$$

$$= 2\pi\eta \frac{\theta}{l} x^2 dx$$

The moment of this force

$$= 2\pi\eta \frac{\theta}{l} x^2 dx \cdot x$$

$$= \frac{2\pi\eta\theta}{l} x^3 \cdot dx$$

Now, integrating between the limits x = 0 and x = r,
We have, total twisting couple on the cylinder

$$= \int_o^r \frac{2\pi\eta\theta}{l} x^3 dx$$

$$= \frac{2\pi\eta\theta}{l} \int_o^r x^3 dx$$

$$= \frac{2\pi\eta\theta}{l} \left[\frac{x^4}{4}\right]_o^r$$

∴ Total twisting couple

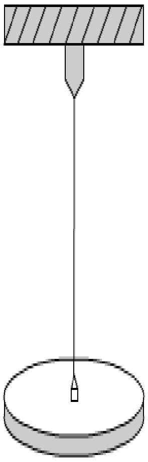
$$= \frac{\pi\eta\theta r^4}{2l} \qquad \text{… … … … … … (3)}$$

Then, the twisting couple per unit twist ($\theta = 1$) is

$$C = \frac{\pi\eta r^4}{2l} \qquad \text{… … … … … … (4)}$$

This twisting couple per unit twist is also called the torsional rigidity of the cylinder or wire.

## ❖ Torsional Pendulum :



A heavy cylindrical rod or disc, suspended from the end of a fine wire, whose upper end is fixed, is called torsional pendulum.

The rod or disc is turned, the wire will twist and when released, it execute torsional vibrations about the axis.

Let $\theta$ be the twisting angle. Then the restoring couple set up in it is.

$$C\theta = \frac{\pi\eta\theta r^4}{2l} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

This produces an angular acceleration $\frac{d\omega}{dt}$ in the rod or the disc

$$\therefore I\frac{d\omega}{dt} = -C\theta \qquad (\because \tau = I\alpha)$$

$$\therefore \frac{d\omega}{dt} = -\left(\frac{C}{I}\right)\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

(- sign indicates the restoring couple)

Where, I is the moment of inertia of the rod or disc.

The motion of the rod or disc is simple harmonic. Its time period is given by

$$t = 2\pi\sqrt{\frac{\text{displacement}}{\text{angular acceleration}}}$$

$$= 2\pi\sqrt{\frac{\theta}{\left(\frac{C}{I}\right)\theta}}$$

$$t = 2\pi\sqrt{\frac{I}{C}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

This is called the equation of time period for torsional pendulum.

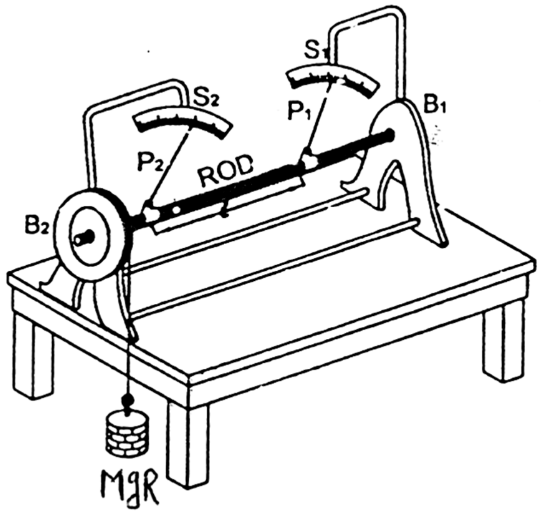
## ❖ Determination of the coefficient of rigidity ($\eta$) for a write :

### (1) Statical Method :

This method is based on the direct application of the expression for the twisting couple

$$c = \frac{\pi\eta r^4}{2l}$$

## ➢ Horizontal twisting apparatus for a rod :




The arrangement of the apparatus is shown in fig. A rod of 50 cm in length and about 0.25 cm in radius is fixed at one end with block $B_1$. A large pulley $B_2$ attached to the other end of the rod. A cord is wound round the pulley and mass M suspended at the other end. Hence couple acts on the rod and twisting produced in it. The Pointer $P_1$ and $P_2$ are clamped on the rod a known distance 'l'. The twisting produced can be measured with the help of scale $S_1$ and $S_2$.

If R is the radius of the pulley, than couple acting on the rod is MgR, where M is mass suspended. This couple is balanced by the torsional couple due to rod and is equal to

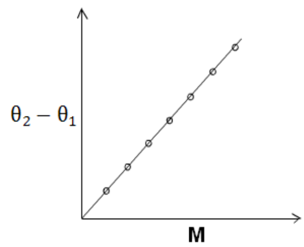
$$= \frac{\pi \eta r^4 (\theta_2 - \theta_1)}{2l}$$

Where, r is the radius of rod, and $\theta_1$ and $\theta_2$ are the angles of twist produced at the two pointers.

Equating the two couples, we have

$$\frac{\pi \eta r^4 (\theta_2 - \theta_1)}{2l} = MgR$$

$$\therefore \ \eta = \frac{2MgRl}{\pi r^4 (\theta_2 - \theta_1)}$$




The experiment is repeated with different masses and a graph is plotted between M and the twist $(\theta_2 - \theta_1)$. The slope of the straight line gives the mean of $\left(\frac{\theta_2 - \theta_1}{M}\right)$ which is used in the above expressions to find out $\eta$.

## ➢ Draw backs of the statical method :

(a) There being a pointer moving over the circular scale, an error is caused due to the eccentricity of the axis of the rod with respect to it.

(b) Since the force is applied through the pulley, a side pull is produced on the rod. This results in friction in the bearings which opposes the rod from twisting freely.

## (2) Dynamical Method:

In this method, a disc or rod, which is suspended from the wire and performing torsional vibrations about the wire and the time period of a body is determined.

Maxwell derived a method using which we can easily determined the moment of inertia of a body without knowing the couple per unit twist.

## ❖ Maxwell's Vibrating needle method :

A hollow tube, open at both ends is suspended at the middle with the torsion wire whose modulus of rigidity is to be measured.

It is suspended vertically from a support and a small piece of mirror attached to it, as shown in figures.

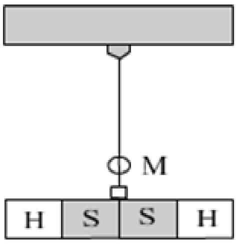



Figure : (a)

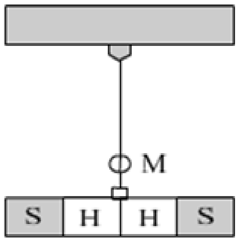



Figure : (b)

As shown in fig.(a), two hollow and two solid cylinder of equal length fitted into tube end to end. The solid cylinders are first into the inner position and hollow cylinders in the outer position as shown in fig.(a). The time period of a given system is given by

$$t_1 = 2\pi\sqrt{\frac{I_1}{C}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where $I_1$ is the moment of inertia of the loaded tube and C is couple per unit twist of the wire.

Now, the position of hollow and solid cylinder are interchanged as shown in fig.(b), Then ,time period $t_2$ of second adjustment is given by

$$t_2 = 2\pi\sqrt{\frac{I_2}{C}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where, $I_2$ is the moment of inertia of the tube in new position.

Squaring equation (1) and (2) , we set

$$t_1^2 = \frac{4\pi^2 I_1}{C} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

And

$$t_2^2 = \frac{4\pi^2 I_2}{C} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Subtracting (3) from (4), we have

$$t_2^2 - t_1^2 = \frac{4\pi^2}{C}(I_2 - I_1) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Now, let $m_1$ be the mass of each hollow cylinder and $m_2$ be the mass of each solid cylinder.

Let the length of tube be 2a. Hence, the length of each solid or hollow cylinder is $a/2$. Then the centers of mass of the inner and outer cylinders lie at distance $a/4$ and $3a/4$.

Hence, in changing from first to second position an extra mass $(m_2 - m_1)$ transferred from a distance $a/4$ to $3a/4$. Then using the principal of parallel axes, we have

$$I_2 = I_1 + 2(m_2 - m_1)\left[\left(\frac{3a}{4}\right)^2 - \left(\frac{a}{4}\right)^2\right]$$

$$= I_1 + 2(m_2 - m_1)\left[\frac{9a^2}{16} - \frac{a^2}{16}\right]$$

$$= I_1 + 2(m_2 - m_1)\frac{a^2}{2}$$

$$I_2 - I_1 = (m_2 - m_1)a^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Substituting this value of $I_2 - I_1$ in equation (5) we get

$$t_2^2 - t_1^2 = \frac{4\pi^2}{C}(m_2 - m_1)a^2$$

$$= \frac{4\pi^2}{\pi\eta r^4/2l}(m_2 - m_1)a^2$$

$$= \frac{8l\pi^2 a^2}{\pi\eta r^4}(m_2 - m_1)$$

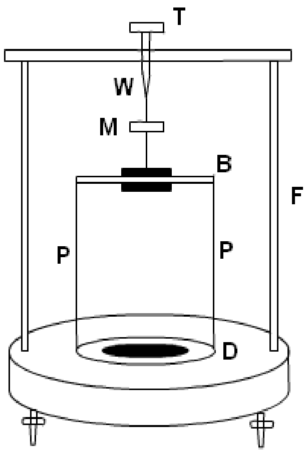
$$\eta = \frac{8\pi l a^2 (m_2 - m_1)}{r^4(t_2^2 - t_1^2)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Thus, if we know l, a, $m_1$, $m_2$, $t_1$, $t_2$, and r, the modulus of rigidity ($\eta$) of the wire can be determined.

## ➢ Advantage :

1. The total suspended mass from the wire remains same, hence value of C remains unchanged.
2. There is no need to find the moment of inertia of the system, hence the question of uncertainty does not arise.

## ❖ Determination of moment of inertia with the help of a torsional pendulum :




The apparatus used here is called the inertia table. As shown in figure. It consists of a horizontal aluminum disc D about 15 cm in diameter, which is fitted with pair of small vertical pillars P with cross bar B. The whole assembly is suspended by a thin wire W from torsion head T inside frame F. The frame is mounted on a heavy iron base. A small piece of mirror M fixed on the cross bar B. The entire apparatus is enclosed in a glass cover.

The disc D is set into torsional vibrations and its time period $t_0$ is measured. If $I_0$ is the moment of inertia of the inertial table and C is the twisting couple per unit twist of the wire then,

$$t_0 = 2\pi\sqrt{\frac{I_0}{C}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The object whose moment of inertia I is to be determined is now placed centrally on the inertia table and its time period $t_1$ is measured.

$$t_1 = 2\pi\sqrt{\frac{(I_0 + I)}{C}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The given body is replaced by an object of a known moment of inertia ($I_1$) and the time period $t_2$ is measured

$$\therefore\ t_2 = 2\pi\sqrt{\frac{(I_0 + I_1)}{C}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now squaring equation (1) and (2) we get

$$t_0^2 = \frac{4\pi^2 I_0}{C} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$t_1^2 = \frac{4\pi^2 (I_0 + I)}{C} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Dividing equation (5) by (4)

$$\frac{t_1^2}{t_0^2} = \frac{I_0 + I}{I_0} = 1 + \frac{I}{I_0}$$

$$\therefore\ \frac{I}{I_0} = \frac{t_1^2}{t_0^2} - 1$$

$$\therefore \frac{I}{I_0} = \frac{t_1^2 - t_0^2}{t_0^2} \qquad \dots\dots\dots\dots\dots\dots (6)$$

Now squaring equation (3) we have

$$t_2^2 = \frac{4\pi^2(I_0 + I_1)}{C} \qquad \dots\dots\dots\dots\dots\dots (7)$$

Dividing equation (7) by (4), we set

$$\frac{t_2^2}{t_0^2} = \frac{I_0 + I_1}{I_0} = 1 + \frac{I_1}{I_0}$$

$$\therefore \frac{I_1}{I_0} = \frac{t_2^2}{t_0^2} - 1$$

$$\therefore \frac{I_1}{I_0} = \frac{t_2^2 - t_0^2}{t_0^2} \qquad \dots\dots\dots\dots\dots\dots (8)$$

Now, dividing equation (6) by (8), we have

$$\frac{{}^{I}/_{I_0}}{{}^{I_1}/_{I_0}} = \frac{t_1^2 - t_0^2}{t_0^2} \text{ X } \frac{t_0^2}{t_2^2 - t_0^2} = \frac{t_1^2 - t_0^2}{t_2^2 - t_0^2}$$

$$\therefore \frac{I}{I_1} = \frac{t_1^2 - t_0^2}{t_2^2 - t_0^2}$$

$$\therefore I = I_1 \text{ x } \frac{t_1^2 - t_0^2}{t_2^2 - t_0^2} \qquad \dots\dots\dots\dots\dots\dots (9)$$

Using above equation, we can determine the moment of inertia of unknown object.

## ❖ Bending of Beams :

A beam is a rod or a bar of uniform cross-section of a homogeneous, isotropic elastic material whose length is very large compared to its thickness.

When a beam is fixed at one end and loaded at the other end as shown in figure (a) within the elastic limit, it will bend and couple produced inside it. The upper surface of the beam gets stretched and becomes a convex shape and lower surface gets compressed and becomes a concave form.

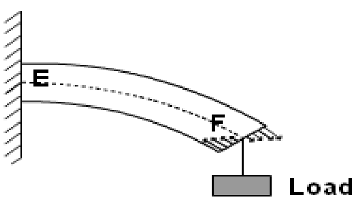



Figure : (a)

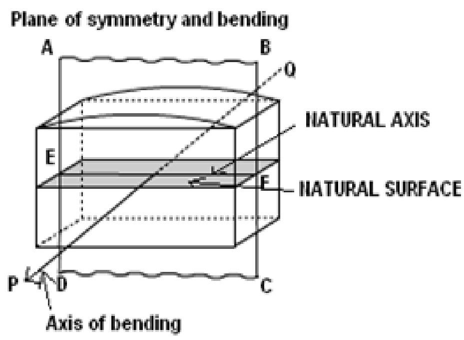



Figure : (B)

All the longitudinal filaments in its upper half are extended and those in lower half are compressed. The extension is maximum in the uppermost filament and the compression is maximum in the lower most filaments. The amount of extension and compression decreases towards the axis of the beam. Thus filament EF neither extended nor compressed. This surface is called Neutral surface. The plane in which all filaments are bent to form circular arcs is called the plane of bending. Thus in figure (b), plane ABCD is the plane of bending. The line perpendicular to the plane of bending is called the axis of bending. Thus, line EF is the neutral axis.

## ❖ Bending Moment :

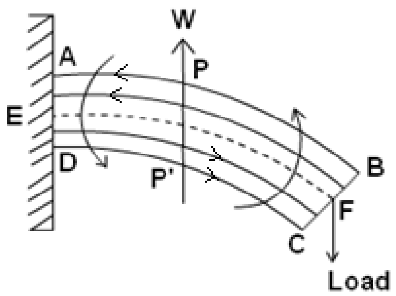



Figure : (a)

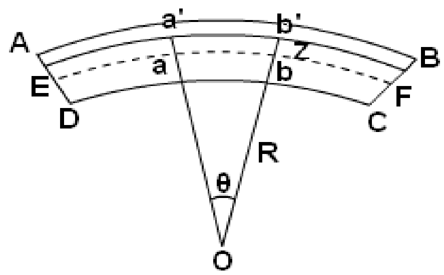



Figure : (b)

Let a beam AB be fixed at A and loaded at B as shown in figure (a). EF is the neutral axis of the beam.

Let us consider a section PBCP’ cut by a plane PP’ at right angles to its length. An equal and opposite reactional force W must be acting vertically upward direction along PP’. The beam bend or rotate in clock wise direction. The couple produced in the beam due to the load applied to the free end of the beam is called the **bending couple** and the moment of this couple is called the **bending moment**.

Let a small part of the beam bent in the form of a circular arc as shown in figure (b). This are subtending angle $\theta$ at 0. Let R be the radius of curvature of this part of the neutral axis. Let a’b’ be an element at a distance Z from the neutral axis.

We know that , arc = Radius X angle subtended

$$\therefore\ a'b' = (R + Z) \cdot \theta$$

$$\text{The original length ab} = R \cdot \theta$$

$$\therefore \text{ increase in length} = a'b' - ab$$

$$= (R + Z) \cdot \theta - R \cdot \theta$$

$$= Z \cdot \theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{Now, strain} = \frac{\text{Change in length}}{\text{Original length}}$$

$$= \frac{Z \cdot \theta}{R \cdot \theta} \qquad = \frac{Z}{R} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Hence, the strain is proportional to the distance from the natural axis.

Now, consider a small area δa at a distance Z from the natural axis.

Young's modulus

$$Y = \frac{\text{Stress}}{\text{Strain}}$$

$$\therefore \text{Stress} = Y \text{ x Strain}$$

$$\therefore \frac{F}{\delta a} = Y \text{ x } \frac{Z}{R}$$

$$\therefore \textit{The force F on area } \delta a = Y \text{ x } \frac{Z}{R} \text{ x } \delta a \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Then, the moment of this force = Force x distance

$$= Y \text{ x } \frac{Z}{R} \text{ x } \delta a \text{ x } Z$$

$$= Y \text{ x } \frac{Z^2}{R} \text{ x } \delta a \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Then, the total moment of forces acting on all the filament is given by

$$\sum \frac{Y \cdot \delta a \cdot Z^2}{R} = \frac{Y}{R} \sum \delta a \cdot Z^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Here, $\sum \delta a \cdot Z^2$ is called the geometrical moment of inertia $I_g$ of the section.

$$\therefore I_g = \sum \delta a \cdot Z^2 = ak^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Where 'a' is the area of the surface and 'k' is the radius of gyration.

$$\therefore \text{The moment of forces} = \frac{Y}{R} \cdot ak^2$$

$$= \frac{Y}{R} \cdot Ig \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

This is called the restoring couple or the bending moment of the beam.

$\therefore$ The bending moment M of beam is

$$M = \frac{Y}{R} \cdot Ig \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Here, the quantity $Y \cdot I_g = Y \cdot ak^2$ is called the flexural rigidity of the beam.

- For rectangular cross section, $a = b \text{ x } d$, and $k^2 = \frac{d^2}{12}$

$$\therefore\ I_g = ak^2 = bd \text{ x } \frac{d^2}{12} = \frac{bd^3}{12}$$

∴ The bending moment for rectangular cross section

$$M = \frac{Ybd^3}{12R} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

- For circular cross section, $a = \pi r^2$ and $k^2 = \frac{r^2}{4}$

$$\therefore\ I_g = ak^2 = \pi r^2 \text{ x } \frac{r^2}{4} = \frac{\pi r^4}{4}$$

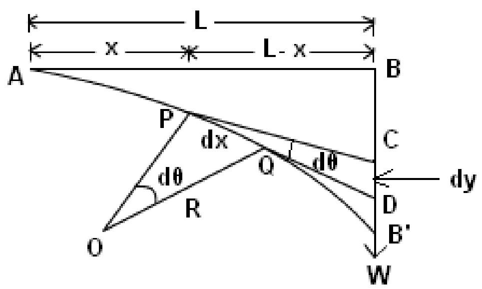
$$\therefore\ M = \frac{Y \cdot \pi r^4}{4R} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

From above equation it is clear that, the bending moment M is directly proportional to the Young's modulus (Y) of the beam.

## ❖ The cantilever :

A beam fixed horizontally at one end and loaded at the other end is called cantilever.

- **When the weight of the beam is ineffective :**




Let AB be the natural axis of the cantilever of length L as shown in Figure. It is fixed at end and loaded at B with a weight W. Then the end B is depressed into the position B' and the natural axis takes up the position AB'. Consider a section P of the beam at a distance 'x' from the fixed end A.

The bending moment = W x PB'

$= W(L - x)$

Since the beam is in equilibrium.

∴ We can write

$$W(L - x) = \frac{YI_g}{R} = \frac{Yak^2}{R} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where, R is the radius of curvature.

Thus, for point Q at a small distance dx from P, we have

$$PQ = R \cdot d\theta$$

$$\therefore dx = R \cdot d\theta$$

$$\therefore R = \frac{dx}{d\theta}$$

∴ Equation (1) becomes,

$$W\,(L-x) = \frac{Y.\,ak^2.\,d\theta}{dx}$$

$$\therefore\ d\theta = \frac{W\,(L-x).\,dx}{Y.\,ak^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now, the depression of Q below P is equal to CD or equal to $dy$, then

$$dy = (L-x)d\theta$$

$$= \frac{(L-x)\cdot W(L-x)\cdot dx}{Yak^2}$$

$$dy = \frac{W\cdot(L-x)^2\cdot dx}{Yak^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now, the total depression

$$y = \int_O^L dy = \int_O^L \frac{W\,(L-x)^2\,dx}{Y\cdot ak^2}$$

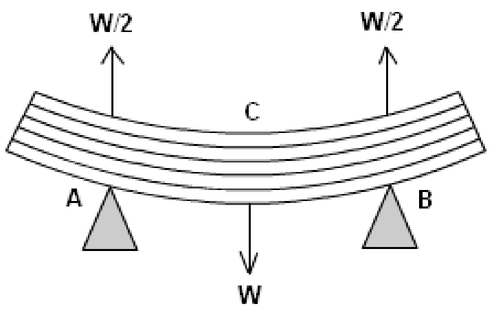
$$= \frac{W}{Y\cdot ak^2}\int_O^L (L^2 - 2Lx + X^2)dx$$

$$= \frac{W}{Y\cdot ak^2}\left[L^2x - 2L\frac{x^2}{2} + \frac{x^3}{3}\right]_O^L$$

$$= \frac{W}{Y\cdot ak^2}\left[L^3 - L^3 + \frac{L^3}{3}\right]$$

$$\therefore\ y = \frac{WL^3}{3Yak^2} = \frac{WL^3}{3YI_g} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

## ❖ Depression of a beam supported at the ends (When the beam is loaded at the center):




Let a beam be supported on two knife edges at its two ends A and B, and let it be loaded in the middle at C with weight W as shown in figure.

Since the middle part of the beam is horizontal, then the beam may be considered as equivalent to two inverted cantilevers, fixed at C and loaded at ends A and B by weight $W/2$.

If L is the length of the beam AB, then the length of each cantilever is $L/2$.

Then the depression of C below A and B is given by

$$y = \frac{(W/2) \cdot (L/2)^3}{3 \cdot Y \cdot ak^2}$$

$$\therefore y = \frac{WL^3}{48\, Y \cdot ak^2} = \frac{WL^3}{48\, Y \cdot I_g} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If the beam is having a circular cross section,

$$\text{then, } I_g = ak^2 = \frac{\pi r^4}{4}$$

$$y = \frac{WL^3}{48 \cdot Y} \cdot \frac{4}{\pi r^4} = \frac{WL^3}{12\, \pi r^4 \cdot Y} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where, r is the radius of cross section. If the beam is having a rectangular cross section,

$$\text{then, } I_g = ak^2 = \frac{bd^3}{12}$$

$$y = \frac{WL^3}{48 \cdot Y} \cdot \frac{12}{bd^3} = \frac{WL^3}{4Y \cdot bd^3} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

## Solved Numerical

**Ex:1** Calculate the twisting couple on a solid shaft of length 1.5 m and diameter 120 mm when it is twisted through an angle$0.6^o$. The coefficient of rigidity for the material of the shaft may be taken to be 93 x $10^9$ N/$m^2$.

**Sol:**

$$C = \frac{\pi \eta r^2 \theta}{2\, l}$$

Here, $\theta = 0.6^o = \frac{\pi}{180}$ x 0.6 radian

$l$ = 1.5 m

r = $\frac{D}{2}$ = 60 mm = 0.06 m

$\eta$ = 93 x $10^9$ N/$m^2$

$$\therefore\ C = \frac{\pi \times 93\ \times\ 10^9 \times (0.06)^4 \times 0.6\, \pi}{2\ \times 1.5 \times 180}$$

$$\therefore\ C = 1.322\ \times 10^4\ N\, m$$

**Ex: 2** A sphere of mass 0.8 kg and radius 0.03 m is suspended from a wire of length 1 m and radius 5 x $10^{-4}$ m. If the period of torsional oscillations of this system is 1.23 sec. Calculate the modulus of rigidity of the wire.

**Sol:**

Here,

$$t = 2\pi\sqrt{\frac{I}{C}}$$

But

$$I = \frac{2}{5}MR^2 \text{ for sphere}$$

and,

$$C = \frac{\pi\eta r^4}{2l}$$

$$\therefore t = 2\pi\sqrt{\frac{4MR^2 l}{5\pi\eta r^4}}$$

Squaring,

$$t^2 = \frac{16\pi M R^2 l}{5\eta r^4}$$

$$\therefore \eta = \frac{16\pi M R^2 l}{5t^2 r^4}$$

Here, M = 0.8 kg
R = 0.03 m
l = 1 m
r = 5 x $10^{-4}$ m
t = 1.23 sec.

$$\therefore \eta = \frac{16\pi \times 0.8 \times (0.03)^2 \times 1}{5 \times (1.23)^2 \times (5 \times 10^{-4})^4}$$

$$\therefore \eta = 7.654 \times 10^{10}\ N/m^2$$

**Ex: 3** A cylindrical rod of diameter 14 mm rests on two knife – edges 0.8 m apart and a load of 1 kg is suspended from its mid-point. Neglecting the weight of the rod, calculate the depression of the mid-point if Y for its material be 2.04 x $10^{11}$ N / $m^2$.

**Sol:**

We know that the depression y of the mid-point of a beam of circular cross-section, supported at the ends loaded in the middle is given by

$$y = \frac{WL^3}{12\pi r^4 \cdot Y}$$

Here, L = 0.8 m,

r = 0.014/2 = 0.007 m,

W = 1 x 9.81 N

And, Y = 2.04 x $10^{11}$ N/$m^2$

So that the depression of the mid-point of the beam is given by

$$y = \frac{9.81 \times (0.8)^3}{12 \times \pi \times (0.007)^4 \times 2.04 \times 10^{11}} = 0.000272\, m = 0.272\, mm$$

**Ex: 4** A brass bar 1 cm square in cross section is supported on two knife edge 100 cm apart. A load of 1 kg at the center of the bar depresses that point by 2.51 mm. What is Young's modulus for brass?

**Sol:** We know that the depression of the mid – point of the bar is given by

$$\mathrm{y} = \frac{\mathrm{WL^3}}{48\,\mathrm{Y} \cdot \mathrm{I_g}}$$

Now, for a bar of rectangular cross – section, $I_g = b\, d^3 / 12$

Here, b = d = 1 cm, $\therefore b\, d^3 = 1$

W = mg= 1000 x 981 dynes

L = 100 cm

y = 2.51 mm = 0.251 cm

Therefore,

$$y = \frac{WL^3}{48\, Y \times b\, d^3 / 12} = \frac{WL^3}{4\, Y \times b\, d^3}$$

$$\therefore Y = \frac{WL^3}{4\, y \times b\, d^3} = \frac{981 \times 10^9}{4 \times 0.251} = 9.77 \times 10^{11} dynes/cm^2$$

**Ex:5** A square metal bar of 2.51 cm side, 37.95 cm long, and weighing 826 gm is suspended by a wire 37.85 cm long and 0.0501 cm radius. It is observed to make 50 compete swings in 335.7 sec. What is the rigidity coefficient of the wire ?

**Sol:**

Here, time period of the bar

t = 335.7 / 50 = 6.714 sec.

L=37.95 cm, B= 2.51 cm , M=826 gm, r=0.0501 cm and l=37.85 cm

Now, time period of a body executing a torsional vibration is given by

$$t = 2\pi \sqrt{I/C}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The moment of inertia of rectangular bar is $I = M\left(\frac{L^2+B^2}{12}\right)$

$$\therefore I = mass\left(\frac{length^2 + breadth^2}{12}\right) = 826 \times \frac{(37.95)^2 \times (2.51)^2}{12}$$

$$\therefore I = 826 \times \frac{1446.3}{12} = 99540\ gm\ cm^2$$

Substituting the value of periodic time $t$ and moment of inertia $I$ in equation (1),
We have,

$$\therefore 6.714 = 2\pi\sqrt{99540 / C}$$

Squaring which, we have

$$C = 4\,\pi^2 \times 99540 / (6.714)^2$$

But, $C = \eta\,\pi\,r^4 / 2\,l$

$$\therefore \frac{\eta\,\pi\,r^4}{2\,l} = \frac{4\,\pi^2 \times 99540}{(6.714)^2}$$

$$\therefore \frac{\eta \times \pi(0.0501)^4}{2 \times 37.85} = \frac{4\,\pi^2 \times 99550}{(6.714)^2}$$

Then, coefficient of rigidity,

$$\eta = \frac{8\,\pi \times 99540 \times 37.85}{(0.0501)^4 \times (6.714)^2} = 3.357 \times 10^{11}\ dynes / cm^2$$

## Exercise

(1) A uniform metal disc of diameter 0.1 m and mass 1.2 kg is fixed symmetrically to the lower end of a torsion wire of length 1 m and diameter 1.44 x $10^{-3}$ m whose upper end is fixed. The time period of torsional oscillations is 1.98 sec. Calculate the modulus of the rigidity of the material of the wire.

( Ans. : 3.579 x $10^{10}$ N / $m^2$ )

(2) What couple must be applied to a wire, 1 meter long, 1 mm diameter, in order to twist one end of it through $90^0$, the other end remaining fixed? The rigidity modulus is 2.8 x $10^{11}$dynes $cm^{-2}$.

( Ans. : 4.3 x $10^6$ dynes cm )

(3) What couple must be applied to a wire 1 meter long and 2 mm in diameter in order to twist one of its ends through $45^0$ when the other remains fixed.
Given $\eta$ = 5 x $10^{11}$ dynes / $cm^2$. ( Ans. : 6.1 x $10^5$ dynes cm )

# Question Bank

**Multiple Choice Questions:**

(1) The twisting couple per unit twist of a cylinder depends on __________
(a) Young's modulus (b) Bulk modulus
(c) Modulus of rigidity (d) Poisson's ratio

(2) If the material of a beam is _____, no bending should be produced.
(a) Homogenous (b) Isotropic
(c) Elastic (d) Plastic

(3) The unit of twisting couple is __________
(a) dynes/cm (b) $N \cdot m$
(c) $N^2 \cdot m$ (d) $N \cdot m^2$

(4) On which of the followings the twisting couple per unit twist of a cylinder depends?
(a) Young's modulus (b) Bulk modulus
(c) Modulus of rigidity (d) Poisson's ratio

(5) The time period of the torsional vibrations(pendulum) is given by __________
(a) $t = 2\pi\sqrt{\frac{C}{I}}$ (b) $t = 2\pi\sqrt{\frac{I}{C}}$
(c) $t = 2\pi\sqrt{\frac{I}{Y}}$ (d) $t = 2\pi\sqrt{\frac{C}{K}}$

(6) A rectangular beam is bent into the arc of a circle, the strain produced in the beam is ________
(a) Extensional only (b) Compression only
(c) Both (d) Shearing

(7) In which part of cantilever the extension is maximum?
(a) Lowermost (b) Uppermost
(c) Middle (d) None of these

(8) The line of intersection of the plane of bending with the neutral surface perpendicular to is called the _________
(a) Neutral surface (b) Plane of bending
(c) Neutral axis (d) Axis of bending

(9) The material of a beam should not be __________
(a) Homogenous (b) Isotropic
(c) Elastic (d) Plastic

(10) The bending moment of a beam depends on only ________
(a) Young's modulus (b) Bulk modulus
(c) Modulus of rigidity (d) Poisson's ratio

(11) Let y is depression produced in the free end of cantilever when weight W is loaded at other end of the beam. If the weight is doubled, the depression y will be
(a) y/2 (b) Y
(c) 2y (d) 4y

(12) Let y is depression produced in the free end of cantilever when weight W is loaded at other end of the beam. If the length of the beam is doubled, the depression y will be ______

(a) y/8 (b) 2y

(c) 8y (d) 4y

(13) Let y is depression produced in the free end of cantilever when weight W is loaded at other end of the beam. If the length of the beam is reduced to L/2 and weight W/2, the depression y will be ________

(a) y/16 (b) 8y

(c) 16y (d) 4y

(14) The torsional rigidity of a cylinder is equal to ________

(a) $Y \frac{Ig}{R}$ (b) $Y \frac{R}{Ig}$

(c) Y x R×Ig (d) None of above

(15) The twisting couple per unit twist of wire or cylinder is also called __________

(a) Young Modulus (b) Modulus of rigidity

(c) Bulk Modulus (d) Torsional rigidity

(16) The twisting couple is equal and opposite to the _______

(a) Force (b) Pure shear

(c) Work (d) Restoring couple

(17) The periodic time of torsional pendulum depends on ________

(a) Young Modulus (b) Torsional rigidity

(c) Bulk Modulus (d) Amplitude of the oscillation

(18) The time period of a torsional pendulum is directly proportional to the square root of __________

(a) Distance (b) Vibrations

(c) Moment of inertia (d) Force

(19) The upper end of a wire of radius 4 mm and length 100 cm is clamped and its other end is twisted through an angle of $30^0$. Then angle of shear is _______

(a) $12^0$ (b) $0.12^0$

(c) $1.2^0$ (d) $0.012^0$

(20) The depression produced in the free end of a cantilever is ________

(a) $y = \frac{W2L^3}{3\,Y\,I_g}$ (b) $y = \frac{3WL^3}{Y\,I_g}$

(c) $y = \frac{WL^3}{3\,Y\,I_g}$ (d) $y = \frac{LW^3}{3\,Y\,I_g}$

(21) Mathematical expression of flexural rigidity is ________

(a) $Y^2ak$ (b) $Ya^2k$

(c) $Ya^2k^2$ (d) $Yak^2$

(22) The geometrical moment of inertia is given by _________

(a) $I_g = a^2k$ (b) $I_g = ak^2$

(c) $I_g = a^2/k$ (d) $I_g = k/ a^2$

**Short Questions**

(1) Define torsional pendulum and write the expression of its time period.
(2) Write the expression of torsional rigidity of wire.
(3) What is statical method?
(4) What is dynamical method?
(5) State the drawbacks of statical method of determination of modulus of rigidity.
(6) List the methods of determination of modulus of rigidity of a cylindrical rod or a wire.
(7) Discuss advantages of dynamical method for determination of modulus of rigidity.
(8) Define and explain bending moment.
(9) Define: cantilever and bending of beam.
(10) Explain the basic assumptions for the theory of bending.
(11) What is cantilever? Write expressions for depression of cantilever when the load is fixed at the center for rectangular and circular bar.
(12) Write name of the two experimental methods to determine (i) Young's modulus and (ii) modulus of rigidity.
(13) The upper end of a wire of radius 4 mm and length 100 cm is clamped and its other end is twisted through an angle of $30^0$. Calculate the angle of shear.
(14) Define and discuss the terms ( i) bending of a beam and (ii) Bending moment.
(15) Write only the equations for the depression of the mid-point of rectangular and cylindrical beams loaded at the centre and supported at ends.

## Long Questions:

(1) Derive an expression for torsional rigidity of the cylinder or a rod of uniform circular section.
(2) Derive the equation for the couple per unit twist produced in a cylindrical wire or shaft with the help of necessary figures.
(3) Derive an expression for periodic time of a torsional pendulum $T = 2\,\pi\sqrt{\frac{I}{C}}$.
Discuss applications of torsional pendulum.
(4) Define torsional pendulum and derive the equation for its time period.
(5) Explain the statical method of determination of modulus of rigidity and also mention its drawbacks.
(6) Describe dynamical method for determination of modulus of rigidity. Also discuss advantages of this method.
(7) What is inertia table? With the help of necessary figure explain how it can be used to determine the moment of inertia of an object.
(8) Describe statical method (horizontal twisting apparatus for a rod) of determination of modulus of rigidity. Discuss drawbacks of the method.
(9) Describe dynamical method (Maxwell's vibrating needle method) of determination of modulus of rigidity.
(10) What is cantilever? Derive the equation for the depression produced in the free end of the cantilever if the weight of the beam is ineffective.
(11) What is bending moment? Derive the equation for the bending moment of beams

having rectangular and circular cross-sections.

(12) Derive an expression for the depression of free loaded end neglecting weight of cantilever.

(13) Explain the concept of bending moment on the basis of theory of banding.

(14) What is bending moment? Show that the bending moment of a beam is $M = \frac{Y}{R} I_g$.

(15) Derive an expression for the depression of free loaded end neglecting weight of cantilever.

(16) Prove that the bending moment of beam is directly proportional to the Young Modulus.

(17) Obtain the formula for the depression of a beam supported at the ends and loaded at the centre.

(18) Derive an expression for depression of cantilever, when the load is fixed at the center. Also find the expression for rectangular and circular cross sections.

(19) Derive the expression bending of a tube supported at the 2 ends & loaded in the middle.

## Answer key of MCQ:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| (1) | (c) | (2) | (d) | (3) | (b) | (4) | (c) |
| (5) | (b) | (6) | (c) | (7) | (b) | (8) | (c) |
| (9) | (d) | (10) | (a) | (11) | (a) | (12) | (c) |
| (13) | (a) | (14) | (a) | (15) | (d) | (16) | (d) |
| (17) | (b) | (18) | (c) | (19) | (b) | (20) | (c) |
| (21) | (d) | (22) | (b) | | | | |