UNIT 2

Curve sketching

- **Definition :**

Cartesian equation : Equation in the form $f(x, y) = 0$ or $y = g(x)$ or $x = h(y)$ is called **Cartesian equation.**

Parametric equation : Equation in the form $x = f(t)$; $y = g(t)$ where t is parameter are called parametric equations.

e.g. $x^2 = 4ay$; $x^2 + y^2 = 4$; $y = \dfrac{x^2 - 1}{x^2 + 4}$; $y = \dfrac{(x-2)(x+1)}{x}$ are cartesian equations.

GRAPH OF CARTESIAN EQUATION

To sketch the graph of cartesian equation we have to discuss following points.

(1) Intercepts :

For

x – intercepts : put $y = 0$

y – intercepts : put $x = 0$

(2) Symmetry :

We have to discuss three types of symmetry.

(i) Symmetry about X-axis : Given curve is said to symmetry about X-axis, if we replace y by $-y$ equation remains unchanged.

(ii) Symmetry about Y-axis : Given curve is said to symmetry about Y–axis, if we replace x by $-x$ equation remains unchanged.

(iii) Symmetry about origin : Given curve is said to symmetry about origin, if we replace x by $-x$ and y by $-y$, equation remains unchanged.

- **Remark :**

(i) If all powers of x are even powers, then equation is symmetry about Y-axis.

(ii) If all powers of y are even powers, then equation is symmetry about X-axis.

(iii) If given equation is symmetry about X-axis and Y-axis both then it is also symmetry about origin.

(3) **Asymptote :**

Discuss asymptote for Cartesian curve.

Ans. :

Def^n : A line is said to asymptote of the curve, if the perpendicular distance between points on the curve and points on the line tends to 0, when curve goes away from origin.

- There are two types of asymptotes.

(i) **Vertical asymptotes :** For the curve $y = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. Take $q(x) = 0$ then find values of x.

(ii) **Horizontal asymptotes :** For the curve $y = \dfrac{p(x)}{q(x)} = \dfrac{a_0 + a_1x + a_2x^2 + ... + a_mx^m}{b_0 + b_1x + b_1x^2 + ... + b_nx^n}$.

→ If $m = n$ then $y = \dfrac{a_m}{b_n}$ is horizontal asymptote.

→ If $m < n$, then $y = 0$ is horizontal asymptote.

→ If $m > n$, then horizontal asymptote is not possible.

Ex. 1. Find asymptote for the curve $y = x^3 - 3x^2 - 2x$.

Sol^n. :

- **Vertical asymptote :** Take $q(x) = 0$

  $\therefore$ $1 = 0$, not possible

  $\therefore$ Vertical asymptote is not possible.

- **Horizontal asymptote :** $y = \dfrac{x^3 - 3x^2 - 2x}{1}$

  $m = 3$, $n = 0$, $m > n$

$\therefore$ Horizontal asymptote is not possible.

(4) **Sign of 'y' :** For the curve $y = \dfrac{p(x)}{q(x)}$

- First take $p(x) = 0$ and $q(x) = 0$, then find values of x.
- Arrange above values in increasing order and make different intervals.
- Check sign of y in above intervals.

Ex. 2. Sketch the following curve. OR Trace the graph of following. OR Disscuss symmetry, intercepts, asymptotes and sign of function for the following curve. Hence sketch the curve.

(1) $y = x^3 - 3x^2 + 2x$ **SPU, June-2012, December-2012**

Sol^n. :

$y = x(x^2 - 3x + 2)$

$y = x(x-2)(x-1)$

(i) Intercept :

→ **X – intercept :** Put $y = 0$, we get.

$0 = x(x-2)(x-1)$

$\therefore$ $\boxed{x = 0, 1, 2}$ are x – int.

→ **Y – intercept :** Put $x = 0$, we get.

$\boxed{y = 0}$ is y – int.

(ii) Symmetry :

→ **Symmetry about X-axis :** If we replace y by $-y$, we get

$-y = x(x-2)(x-1)$

$\therefore$ $y = -x(x-2)(x-1)$

Thus equation is change.

$\therefore$ It is not symmetry about x-axis.

→ **Symmetry about Y-axis :** If we replace x by $-x$, we get

$y = -x(-x-2)(-x-1)$

$\therefore$ $y = -x(x+2)(x+1)$

Thus, equation is change.

$\therefore$ It is not symmetry about y–axis.

→ **Symmetry about origin :** If we replace x by $-x$ and y by $-y$. We get,

$-y = -x(-x-2)(-x-1)$

$\therefore$ $-y = -x(x+2)(x+1)$

$\therefore$ $y = x(x+2)(x+1)$

Thus equation is changed.

$\therefore$ It is not symmetry about origin.

(iii) Asymptotes :

Here, $y = \dfrac{x(x-2)(x-1)}{1}$

→ **Vertical Asymptotes :**

Take $1 = 0$, which is not possible, so vertical asymptote is not possible.

→ **Horizontal Asymptotes :**

Here, $m = 3$, $n = 0$, $m > n$.

$\therefore$ Horizontal asymptote is not possible.

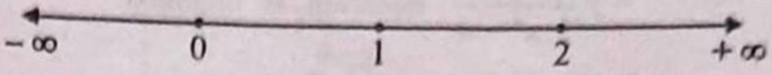
(iv) **Sign of 'y' :**

→ $y = x(x-2)(x-1)$

Take : $x(x-2)(x-1) = 0$

$\therefore \quad x = 0, 2, 1$

$\therefore \quad \boxed{x = 0, 1, 2}$



$\therefore$ Intervals are $(-\infty, 0), (0, 1), (1, 2), (2, \infty)$

Here $y = x(x-2)(x-1)$

If $x \in (-\infty, 0)$, then

$y = (-ve)(-ve)(-ve) \Rightarrow y < 0$

If $x \in (0, 1)$, then

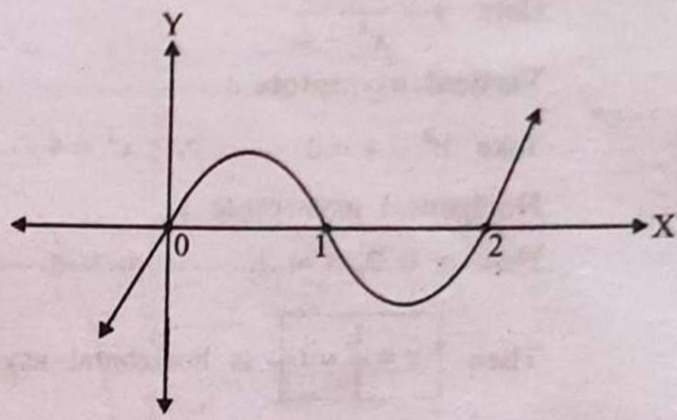
$y = (+ve)(-ve)(-ve) \Rightarrow y > 0$

If $x \in (1, 2)$, then

$y = (+ve)(-ve)(+ve) \Rightarrow y < 0$

If $x \in (2, \infty)$, then

$y = (+ve)(+ve)(+ve) \Rightarrow y > 0$



(2) $y = \dfrac{x^2-1}{x^2-4}$.

SPU, December-2014, June-2011, November-2010

(i) **Intercepts :**

X-intercept : Put $y = 0$

$\therefore \quad 0 = \dfrac{x^2-1}{x^2-4}$

$\therefore \quad 0 = x^2 - 1$

$\therefore \quad \boxed{x = \pm 1}$ are x-int.

Y-intercept : Put $x = 0$

$y = \dfrac{-1}{-4} = \dfrac{1}{4}$

$\therefore \quad \boxed{y = 0.25}$ is y-int.

(ii) **Symmetry :**

→ **Symmetry about X-axis :** If we replace y by $-y$, we get $-y = \dfrac{x^2-1}{x^2-4}$ equation is changed.

$\therefore$ It is not symmetry about x-axis.

→ **Symmetry about Y-axis :** If we replace x by $-x$, we get $y = \dfrac{(-x)^2-1}{(-x)^4-4} = \dfrac{x^2-1}{x^2-4}$, equation is not changed.

$\therefore$ It is symmetry about y-axis.

→ **Symmetry about origin :** If we replace x by $-x$ and y by $-y$, we get

$$-y=\frac{(-x)^2-1}{(-x)^4-4}$$

$$-y=\frac{x^2-1}{x^4-4}$$ equation is changed.

∴ It is not symmetry about origine.

(iii) Asymptotes :

Here $y=\dfrac{x^2-1}{x^2-4}$

Vertical asymptote :

Take $x^2-4=0$ ∴ $x^2=4$ ∴ $\boxed{x=\pm 2}$ are vertical asymptotes.

Horizontal asymptote :

Here $m=2$, $n=2$, ∴ $m=n$

Then $\boxed{y=\frac{1}{1}=1}$ is horizontal asymptote.

(iv) Sign of 'y' :

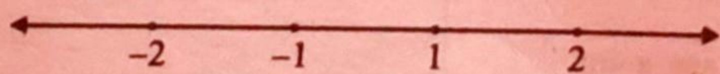
→ $y=\dfrac{x^2-1}{x^2-4}$

Take, $x^2-1=0$ and $x^2-4=0$

∴ $x^2=1$ ∴ $x^2=4$

∴ $x=\pm 1$ ∴ $x=\pm 2$

∴ $\boxed{x=-2,-1,1,2}$



∴ Intervals are $(-\infty,-2)$, $(-2,-1)$, $(-1,1)$, $(1,2)$, $(2,\infty)$

→ $y=\dfrac{x^2-1}{x^2-4}$

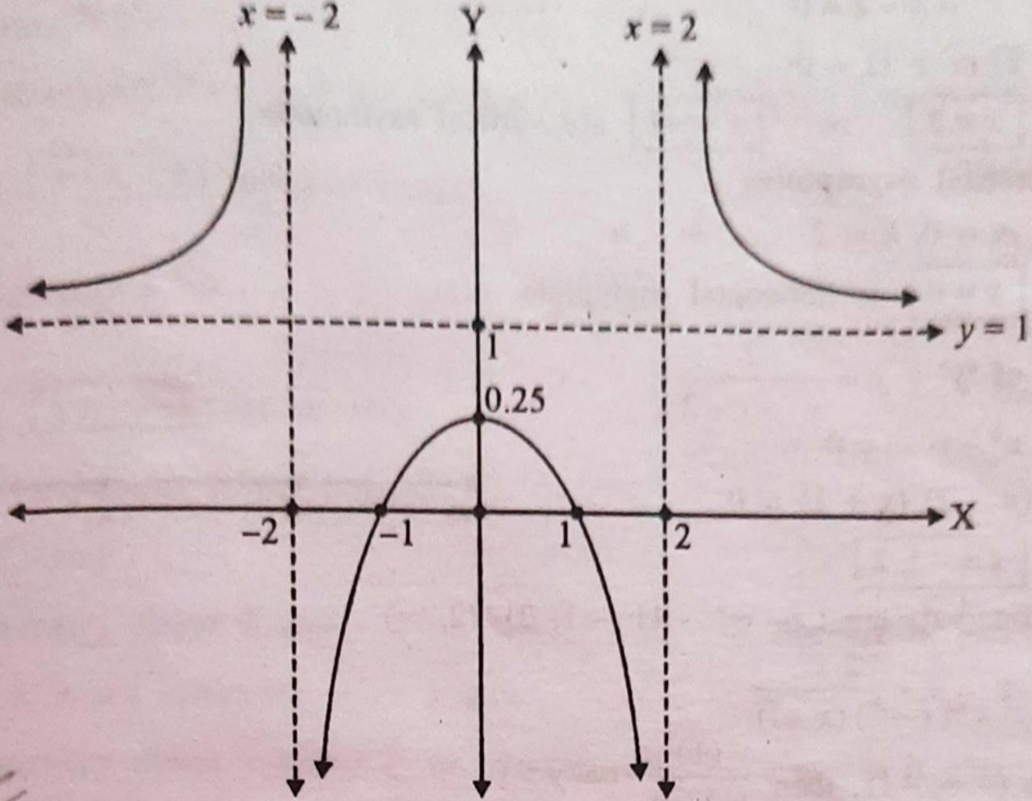
If $x\in(-\infty,-2)$, then $y=\dfrac{(+)}{(+)}\Rightarrow y>0$

If $x\in(-2,-1)$, then $y=\dfrac{(+)}{(-)}\Rightarrow y<0$

If $x\in(-1,1)$, then $y=\dfrac{(-)}{(-)}\Rightarrow y>0$

If $x \in (1, 2)$ then $y = \dfrac{(+)}{(-)} \Rightarrow y < 0$

If $x \in (2, \infty)$ then $y = \dfrac{(+)}{(+)} \Rightarrow y > 0$



(3) $y = \dfrac{2}{(x+1)(x-2)}$ or $y = \dfrac{2}{x^2 - x - 2}$ **SPU, Nov. 2015, April-2015**

Sol^n. :

(i) Intercept :

X-intercept : Put $y = 0$

$\therefore$ $0 = 2$ is not possible.

$\therefore$ X-intercept is not possible.

Y-intercept : Put $x = 0$

$\therefore$ $y = \dfrac{2}{1(-2)} \Rightarrow \boxed{y = -1}$ is Y-int.

(ii) Symmetry :

→ **Symmetry about X-axis :** If we replace y by $-y$, the equation is changed.

$\therefore$ It is not symmetry about X-axis.

→ **Symmetry about Y-axis :** If we replace x by $-x$, the equation is changed.

$\therefore$ It is not symmetry about Y-axis.

→ **Symmetry about origin :** If we replace x by $-x$ and y by $-y$, the equation is changed.

$\therefore$ It is not symmetry about origin.

(iii) **Asymptotes :**

$$y=\frac{2}{x^2-x-2}$$

Vertical asymptotes :

Take $x^2-x-2=0$

$(x-2)(x+1)=0$

$\therefore$ $\boxed{x=2}$ or $\boxed{x=-1}$ are vertical asymptotes.

Horizontal asymptotes :

Here $m=0$, $n=2$ $\therefore$ $m<n$

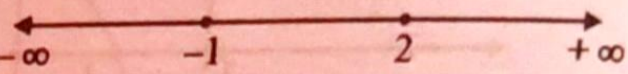
$\therefore$ $\boxed{y=0}$ is horizontal asymptote.

(iv) **Sign of 'y' :** $y=\dfrac{2}{x^2-x-2}$

$\rightarrow$ $x^2-x-2=0$

$(x-2)(x+1)=0$

$\therefore$ $\boxed{x=-1,2}$



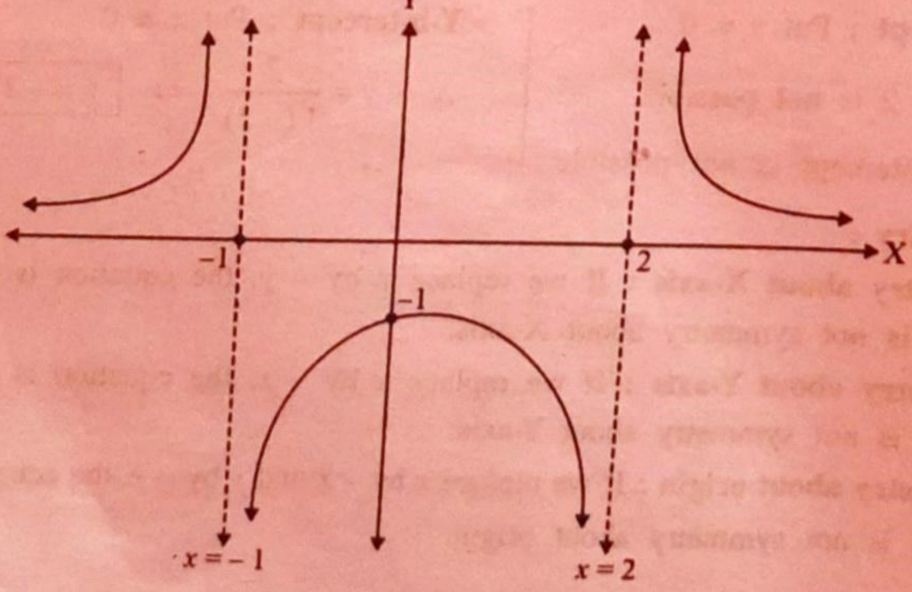
$\therefore$ Intervals are : $(-\infty,-1)$ $(-1,2)$, $(2,\infty)$

Here $y=\dfrac{2}{(x-2)(x+1)}$

If $x\in(-\infty,-1)$, then $\dfrac{(+)}{(-)(-)}$ $\Rightarrow y>0$

If $x\in(-1,2)$, then $y=\dfrac{(+)}{(+)(-)}$ $\Rightarrow y<0$

If $x\in(2,\infty)$, then $y=\dfrac{(+)}{(+)(+)}$ $\Rightarrow y>0$



(4) $y = \dfrac{(x-1)(x+2)}{x(x-4)}$ or $y = \dfrac{x^2+x-2}{x^2-4x}$

SPU, April-2015, November-2012, June-2011, November-2010

Sol^n. :

(i) **Intercept :**

X-intercept : Put $y = 0$, we get $0 = (x-1)(x+2)$

$\therefore$ $\boxed{x = 1, -2}$ are X-intercepts :

Y-intercept : Put $x = 0$, we get $y = \dfrac{(-1)(2)}{0}$

$\therefore$ $\boxed{y = -\infty}$, not possible.

$\therefore$ Y-intercept is not possible.

(ii) **Symmetry :**

Symmetry about X-axis : If we replace y by $-y$, equation is changed.

$\therefore$ It is not symmetry about X-axis.

→ **Symmetry about Y-axis :** If we replace x by $-x$ equation is changed.

$\therefore$ It is not symmetry about Y-axis.

→ **Symmetry about origin :** If we replace x by $-x$ and y by $-y$ equation is changed.

$\therefore$ It is not symmetry about origin.

(iii) **Asymptotes :**

→ Here, $y = \dfrac{(x-1)(x+2)}{x(x-4)}$

Vertical asymptote :

Take $x(x-4) = 0$

$\therefore$ $\boxed{x = 0, 4}$ are vertical asymptotes.

Horizontal asymptote :

Here $m = 2, n = 2$ $\quad\therefore\ m = n$

$\therefore$ $y = \dfrac{1}{1} = \boxed{1}$ is horizontal asymptote.

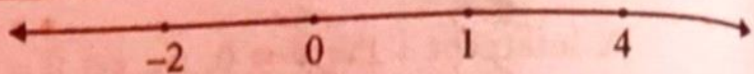
(iv) Sign of 'y' :

Here $y = \dfrac{(x-1)(x+2)}{x(x-4)}$

Take $(x-1)(x+2) = 0$ and $x(x-4) = 0$

$\boxed{x = 1, -2}$ $\therefore$ $\boxed{x = 0, 4}$

$\therefore$ $\boxed{x = -2, 0, 1, 4}$



$\therefore$ Intervals are $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$, $(1, 4)$, $(4, \infty)$.

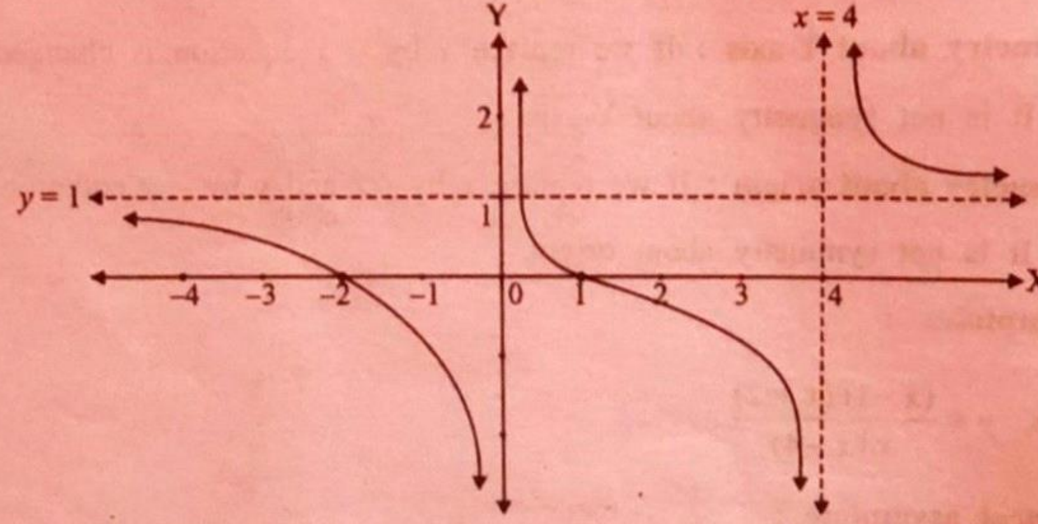
If $x \in (-\infty, -2)$, then $y = \dfrac{(-)(-)}{(-)(-)} \Rightarrow y > 0$

If $x \in (-2, 0)$, then $y = \dfrac{(-)(+)}{(-)(-)} \Rightarrow y < 0$

If $x \in (0, 1)$, then $y = \dfrac{(-)(+)}{(+)(-)} \Rightarrow y > 0$

If $x \in (1, 4)$, then $y = \dfrac{(+)(+)}{(+)(-)} \Rightarrow y < 0$

If $x \in (4, \infty)$, then $y = \dfrac{(+)(+)}{(+)(+)} \Rightarrow y > 0$



(7) $y = \dfrac{(x+3)(x-1)}{x(x+2)} = \dfrac{x^2+2x-3}{x^2+2x}$

Sol^n. :

(i) Intercept :

X-int : Put $y = 0$, we get $0 = (x+3)(x-1)$

$\therefore \quad \boxed{x = -3, 1}$ are x–int.

Y-int : Put $x = 0$, we get $y = \dfrac{(3)(-1)}{0}$

$\therefore \quad \boxed{y = \infty}$ not possible.

$\therefore$ y-intercept is not possible.

(ii) Symmetry :

Symmetry about X-axis : If we replace y by $-y$, equation is changed.

$\therefore$ It is not symmetry about X-axis.

Symmetry about Y-axis : If we replace x by $-x$, equation is changed.

$\therefore$ It is not symmetry about Y-axis.

Symmetry about origin : If we replace x by $-x$ and y by $-y$, equation is changed.

$\therefore$ It is not symmetry about origin.

(iii) Asymptotes :

Here, $y = \dfrac{(x+3)(x-1)}{x(x+2)}$

Take $x(x+2) = 0$

$\therefore \quad \boxed{x = 0, -2}$ are vertical asymptotes.

Also here $m = 2$, $n = 2$ $\quad \therefore \; m = n$

$\therefore \quad y = \dfrac{1}{1} \Rightarrow \boxed{y = 1}$ is horizontal asymptote.

(iv) Sign of 'y' :

Here, $y = \dfrac{(x+3)(x-1)}{x(x+2)}$

Take, $(x+3)(x-1) = 0$ and $x(x+2) = 0$

$\therefore \quad \boxed{x = -3, 1} \qquad \therefore \quad \boxed{x = 0, -2}$

$\therefore \quad \boxed{x = -3, -2, 0, 1}$

$\therefore$ Intervals are $(-\infty, -3)$, $(-3, -2)$, $(-2, 0)$, $(0, 1)$, $(1, \infty)$

Here $y = \dfrac{(x+3)(x-1)}{x(x+2)}$

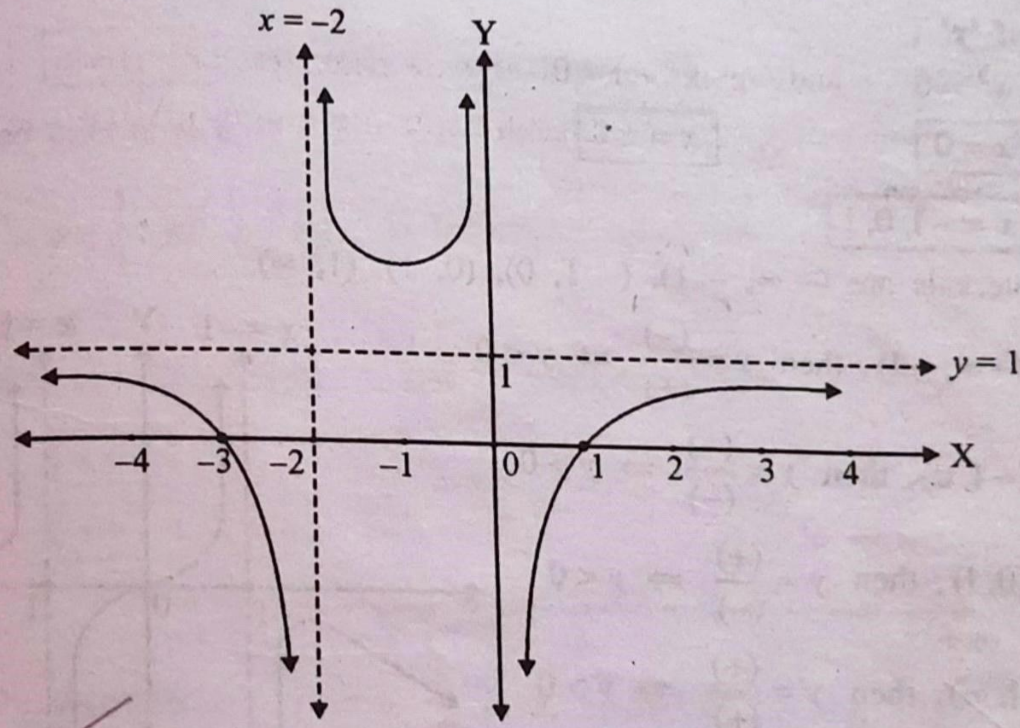
If $x \in (-\infty, -3)$, then $y = \dfrac{(-)(-)}{(-)(-)} \Rightarrow y > 0$

If $x \in (-3, -2)$, then $y = \dfrac{(+)(-)}{(-)(-)} \Rightarrow y < 0$

If $x \in (-2, 0)$, then $y = \dfrac{(+)(-)}{(-)(+)} \Rightarrow y > 0$

If $x \in (0, 1)$, then $y = \dfrac{(+)(-)}{(+)(+)} \Rightarrow y < 0$

If $x \in (1, \infty)$, then $y = \dfrac{(+)(+)}{(+)(+)} \Rightarrow y > 0$



(8) $y = \dfrac{x^3}{x^2 - 1}$

Sol^n. :

(i) **Intercept :**

X-int : Put $y = 0$, we get $0 = x^3$

$\therefore$ $\boxed{x = 0}$ are x–int.

Y-int : Put $x = 0$, we get

$y = \dfrac{0}{0-1} = \dfrac{0}{-1} = \boxed{0}$ is y-int.

(ii) **Symmetry :**

Symmetry about X-axis : If we replace y by $-y$, equation is changed.

$\therefore$ It is not symmetry about X-axis.

Symmetry about Y-axis : If we replace x by $-x$, equation is changed.

$\therefore$ It is not symmetry about Y-axis.

Symmetry about origin : If we replace x by $-x$ and y by $-y$, equation is not changed.

$\therefore$ It is symmetry about origin.

(iii) Asymptotes :

Here, $y = \dfrac{x^3}{x^2 - 1}$

Take $x^2 - 1 = 0$ $\quad \therefore \; x^2 = 1$

$\therefore$ $\boxed{x = \pm 1}$ are vertical asymptotes.

Also here $m = 3, n = 2$ $\quad \therefore \; m > n$

$\therefore$ Horizontal asymptote is not possible.

(iv) Sign of 'y' :

Take, $x^3 = 0$ and $x^2 - 1 = 0$

$\therefore \; \boxed{x = 0}$ $\qquad \therefore \; \boxed{x = \pm 1}$

$\therefore \; \boxed{x = -1, 0, 1}$

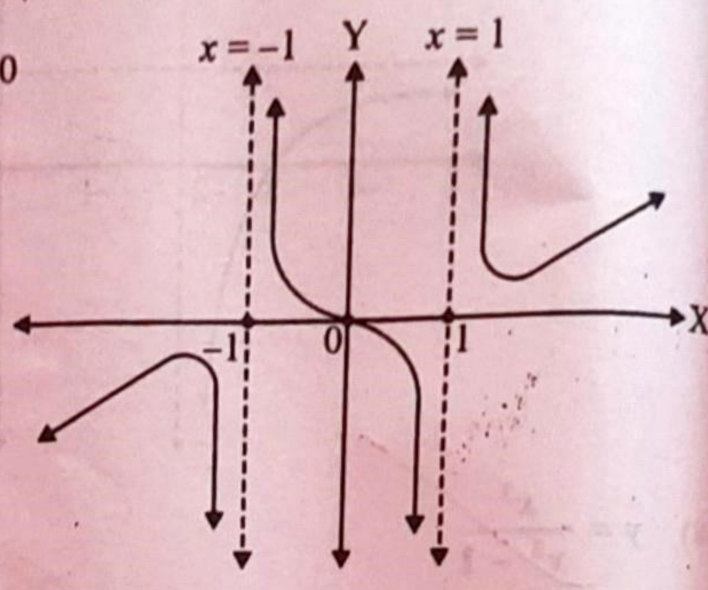
$\therefore$ Intervals are $(-\infty, -1), (-1, 0), (0, 1), (1, \infty)$.

If $x \in (-\infty, -1)$, then $y = \dfrac{(-)}{(+)} \Rightarrow y < 0$

If $x \in (-1, 0)$, then $y = \dfrac{(-)}{(-)} \Rightarrow y > 0$

If $x \in (0, 1)$, then $y = \dfrac{(+)}{(-)} \Rightarrow y < 0$

If $x \in (1, \infty)$, then $y = \dfrac{(+)}{(+)} \Rightarrow y > 0$



(9) $y = \dfrac{(x-2)(x^2+1)}{(x-1)(x+1)^2}$

Sol^n. :

(i) Intercept :

X-int : Put $y = 0$,

$\therefore \; 0 = (x-2)(x^2+1)$

$\therefore \; x - 2 = 0$ or $x^2 + 1 = 0$

$\therefore \; x = 2$ or $x^2 = -1$ not possible.

$\therefore \; \boxed{x = 2}$ is x–intercept.

Y-int : Put $x = 0$

$\therefore \; y = \dfrac{(-2)(1)}{(-1)(1)} = \dfrac{-2}{-1}$

$\therefore \; \boxed{y = 2}$ is y-intercept.

(ii) **Symmetry :**

Symmetry about X-axis : If we replace y by $-y$, equation is changed.

$\therefore$ It is not symmetry about X-axis.

Symmetry about Y-axis : If we replace x by $-x$, equation is changed.

$\therefore$ It is not symmetry about Y-axis.

Symmetry about origin : If we replace x by $-x$ and y by $-y$, equation is changed.

$\therefore$ It is not symmetry about origin.

(iii) **Asymptotes :**

$$y = \frac{(x-2)(x^2+1)}{(x-1)(x+1)^2}$$

Take $(x-1)(x+1)^2 = 0$

$\Rightarrow \quad x-1=0,\ (x+1)^2=0 \quad \Rightarrow x = 1, -1$

$\therefore \quad \boxed{x=1,-1}$ are vertical asymptotes.

Also here $m = 3,\ n = 3 \qquad \therefore\ m = n$

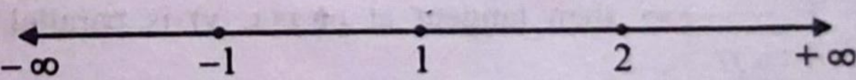
$\therefore \quad y = \frac{1}{1} \Rightarrow \boxed{y=1}$ is horizontal asymptote.

(iv) **Sign of 'y' :**

Take, $(x-2)(x^2+1) = 0$ and $(x-1)(x+1)^2 = 0$

$\therefore \quad \boxed{x=2} \qquad \therefore \quad \boxed{x=1,-1}$

$\therefore \quad \boxed{x=-1,\ 1,\ 2}$



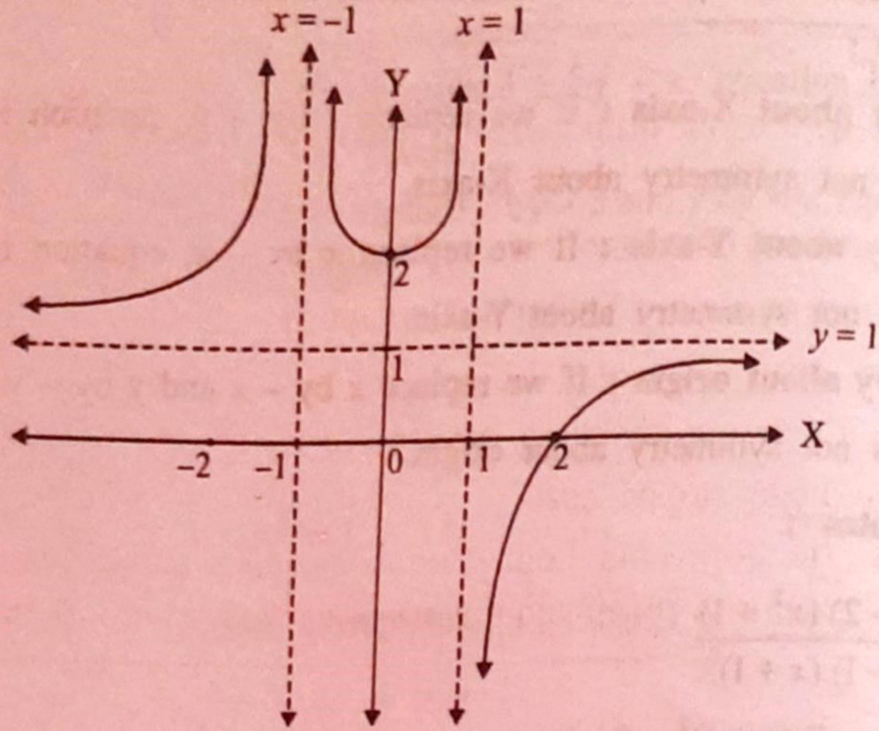
$\therefore$ Intervals are $(-\infty, -1)$, $(-1, 1)$, $(1, 2)$, $(2, \infty)$

If $x \in (-\infty, -1)$, then $y = \frac{(-)(+)}{(-)(+)} \Rightarrow y > 0$

If $x \in (-1, 1)$, then $y = \frac{(-)(+)}{(-)(+)} \Rightarrow y > 0$

If $x \in (1, 2)$, then $y = \frac{(-)(+)}{(+)(+)} \Rightarrow y < 0$

If $x \in (2, \infty)$, then $y = \frac{(+)(+)}{(+)(+)} \Rightarrow y > 0$



GRAPH OF PARAMETRIC EQUATION

To sketch the graph of parametric equation $x = f(t);\ y = g(t)$, we have to discuss following points.

1. **Intercept :** X-intercept and Y-intercept.
2. **Extent to** → of **the curve :** Extent to the curve is region of the curve on X-axis and Y-axis.
3. **Tangent parallel to the** axis **curve :** For the curve $x = f(t);\ y = g(t)$.

   We know that slope of tangent $= \dfrac{dy}{dx}$.

   (1) If $\left(\dfrac{dy}{dx}\right)_{p(x,\,y)} = 0$, then tangent at pt $p(x, y)$ is parallel to X-axis.

   (2) If $\left(\dfrac{dy}{dx}\right)_{p(x,\,y)} \longrightarrow \infty$ then tangent at pt $p(x, y)$ is parallel to Y-axis.

4. **Asymptotes to the curve :** For the curve $x = f(t),\ y = g(t)$. There are two type of asymptotes for parametric curves.

   (i) **Asymptotes parallel to axes :** Find limiting value of parameter of 't' for which any one variable x or y is finite and the other $\to \infty$. Finite value of x and y are called asymptotes parallel to axes.

   (ii) **Oblique asymptotes :** Find limiting value of parameter of 't' for which both variable $x \to \pm\infty$ and $y \to \pm\infty$, then there is a possibility of oblique asymptotes. If oblique asymptote is exists, then it is in the form $\boxed{y = mx + c}$ where $m = \lim\limits_{t} \dfrac{dy}{dx}$, $c = \lim\limits_{t} (y - mx)$.

$\therefore$ Give curve is bounded.

■ **Theorem-1 :**

If a curve given by $x = f(t)$, $y = g(t)$ and both x and y get numerically large as t approches some number say a. Then an oblique asymptote to the curve if it exist is given by $y = mx + c$, where $m = \lim_{t \to a} \left(\frac{dy}{dx}\right)$, $c = \lim_{t \to a} (y - mx)$.

SPU, April-2016, Nov. 2015, April-2015, December-2014, September-2014, November-2013, June-2012, November-2012, December-2012, June-2011

Proof :

We have given that,

If $t \to a$, then $x \to \infty$ and $y \to \infty$.

$\therefore$ By definition, there is a possibility of oblique asymptote.

If it exist then it is in the form $y = mx + c$.

Now we find m :

We know that asymptote becomes a tangent to the curve at infinity. We know that

$$\begin{aligned} m &= \text{slope of tangent to the curve.} \\ &= \text{slope of asymptote at the infinity} \\ &= \frac{dy}{dx} \text{ at infinity} \\ &= \frac{dy}{dx} \text{ at } \begin{matrix} x \to \infty, \\ y \to \infty \end{matrix} \\ &= \frac{dy}{dx} \text{ if } t \to a \end{aligned}$$

$$\therefore \quad \boxed{m = \lim_{t \to a} \left(\frac{dy}{dx}\right)}$$

→ **Now we find c :**

We know that perpendicular distance between any point $p(x, y)$ of the curve to the line $mx - y + c = 0$ is $\left|\frac{mx - y + c}{\sqrt{m^2 + 1}}\right|$

We know that distance

$$\frac{|mx - y + c|}{\sqrt{m^2 + 1}} \to 0 \text{ at infinity i.e., at } x \to \infty \text{ and } y \to \infty$$

$$\text{Thus } \frac{|mx - y + c|}{\sqrt{m^2 + 1}} \to 0, \text{ if } t \to a$$

$$\text{Thus } \lim_{t \to a} \frac{|mx - y + c|}{\sqrt{m^2 + 1}} = 0$$

$$\Rightarrow \lim_{t \to a} |mx - y + c| = 0$$

$$\Rightarrow \lim_{t \to a} mx - y + c = 0$$

$$\Rightarrow \boxed{c = \lim_{t \to a} y - mx}$$

Ex. 4 : Find asymptotes for the curve given by $x = t + \frac{1}{t^2}$; $y = t - \frac{1}{t^2}$.

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Sol^n. :

Asymptote parallel to axes :

Here $x = t + \frac{1}{t^2}$, $y = t - \frac{1}{t^2}$, $t \in R$

Here, we can not find any value of 't' for which one variable x or y is finite and other is infinite.

$\therefore$ Asymptote parallel to axes are not possible.

→ **Oblique asymptotes :**

If $t \to 0$, then $x \to \infty$ and $y \to \infty$

Also, If $t \to \infty$, then $x \to \infty$ and $y \to \infty$

$\therefore$ There is a possibility of oblique asymptote.

If it exist then it is in the form $\boxed{y = mx + c}$ .... (I)

→ **For $t \to 0$:**

We know that $\frac{dx}{dt} = 1 - \frac{2}{t^3} = 1 - 2t^{-3}$ and $\frac{dy}{dt} = 1 + \frac{2}{t^3} = 1 + 2t^{-3}$

$$\therefore \quad \frac{dy}{dx} = \frac{1 + 2t^{-3}}{1 - 2t^{-3}} = \frac{t^3 + 2}{t^3 - 2}$$

$$\therefore \quad m = \lim_{t \to 0} \frac{dy}{dx}$$

$$= \lim_{t \to 0} \left(\frac{t^3 + 2}{t^3 - 2} \right)$$

$$= \frac{0 + 2}{0 - 2}$$

$$\therefore \quad \boxed{m = -1}$$

Also, $c = \lim_{t \to 0} (y - mx)$

$$= \lim_{t \to 0} (y + x)$$

$$c = \lim_{t \to 0} \left(t - \frac{1}{t^2} + t + \frac{1}{t^2}\right)$$

$$= \lim_{t \to 0} (2t)$$

$\therefore$ $\boxed{c = 0}$

By equation (1)

$y = -x + 0$

$\therefore$ $\boxed{y = -x}$ is oblique asymptote.

→ **For $t \to \infty$:**

$$m = \lim_{t \to \infty} \frac{dy}{dx}$$

$$= \lim_{t \to \infty} \left(\frac{t^3 + 2}{t^3 - 2}\right)$$

$$= \lim_{t \to \infty} \left(\frac{1 + \frac{2}{t^3}}{1 - \frac{2}{t^3}}\right)$$

$$= \frac{1+0}{1-0} = 1$$

$\therefore$ $\boxed{m = 1}$

Also, $c = \lim_{t \to \infty} (y - mx) = \lim_{t \to \infty} (y - x)$

$$= \lim_{t \to \infty} \left(t - \frac{1}{t^2} - t - \frac{1}{t^2}\right)$$

$$= \lim_{t \to \infty} \frac{-2}{t^2}$$

$$= 0$$

$\therefore$ $\boxed{c = 0}$

$\Rightarrow$ $\boxed{y = x}$ is oblique asymptote.

Hence $\boxed{y = \pm x}$ are oblique asymptotes.

Which is parametric equation of parabola.

Ex. 6 : Find parametric equation of following.

(1) $x^{2/3} + y^{2/3} = a^{2/3}$

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$$\Rightarrow \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$$

$$\Rightarrow \left[\left(\frac{x}{a}\right)^{1/3}\right]^2 + \left[\left(\frac{y}{a}\right)^{1/3}\right]^2 = 1 \qquad \text{---- (1)}$$

Let $\left(\frac{x}{a}\right)^{1/3} = \cos\theta$, $\left(\frac{y}{a}\right)^{1/3} = \sin\theta$, which satisfies equation (1)

Then $\boxed{x = a\cos^3\theta,\ y = a\sin^3\theta}$

(2) $\sqrt{x} + \sqrt{y} = \sqrt{a}$

SPU, Nov.-2015, Sept. 2014, Nov. 2013

$$\Rightarrow x^{1/2} + y^{1/2} = a^{1/2}$$

$$\therefore \left(\frac{x}{a}\right)^{1/2} + \left(\frac{y}{a}\right)^{1/2} = 1$$

$$\therefore \left[\left(\frac{x}{a}\right)^{1/4}\right]^2 + \left[\left(\frac{y}{a}\right)^{1/4}\right]^2 = 1 \qquad \text{---- (2)}$$

Let, $\left(\frac{x}{a}\right)^{1/4} = \cos\theta$, $\left(\frac{y}{a}\right)^{1/4} = \sin\theta$, which satisfies equation (2).

$$\therefore \frac{x}{a} = \cos^4\theta, \qquad \frac{y}{a} = \sin^4\theta$$

$$\therefore \boxed{x = a\cos^4\theta, \qquad y = a\sin^4\theta}$$

2. $x = \cos^2\theta$, $y = 2\sin\theta$

(1) **Intercept :**

x-intercept : put $y = 0$, we get

$$0 = 2\sin\theta$$

$\therefore \quad \sin\theta = 0$

$\therefore \quad \theta = n\pi,\ n \in Z$

Now, $x = \cos^2\theta = \cos^2 (n\pi)$

$\therefore \quad \boxed{x = 1}$

So, $x = 1$ is x-intercept.

y-intercept put : $x = 0$, we get

$$0 = \cos^2\theta$$

$$\therefore \quad \theta = (2n+1)\frac{\pi}{2},\ n \in Z$$

$$\text{So } y = 2\sin(2n+1)\frac{\pi}{2} = \pm 2$$

$$\Rightarrow \quad \boxed{y = \pm 2}$$

So, $y = \pm 2$ are y-intercept.

(2) **Extent :** We know that

$$-1 \le \cos\theta \le 1$$

$$\therefore \quad 0 \le \cos^2\theta \le 1$$

$$\therefore \quad \boxed{0 \le x \le 1}$$

$$-1 \le \sin\theta \le 1$$

$$\Rightarrow \quad -2 \le 2\sin\theta \le 2$$

$$\therefore \quad \boxed{-2 \le y \le 2}$$

(3) **Tangent parallel to axes :**

$$\frac{dy}{dx} = \frac{2\cos\theta}{-2\cos\theta\sin\theta} = \frac{-1}{\sin\theta}$$

Tangent parallel to x-axis : we know that

$$\frac{dy}{dx} = 0, \text{ if } \frac{1}{\sin\theta} = 0, \quad \text{i.e. } \boxed{1 = 0} \text{ not possible}$$

So, tangent parallel to x-axis is not possible.

Tangent paralle to y-axis : We know that

$$\frac{dy}{dx} \to \infty, \text{ if } \frac{1}{\sin\theta} \to \infty,$$

if $\sin\theta = 0$, if $\theta = n\pi$, $n \in Z$

$$\text{So} \quad x = \cos^2\theta = \cos^2(n\pi)$$

$$\therefore \quad \boxed{x = 1}$$

$$y = 2\sin\theta = 2\sin(n\pi)$$

$$\therefore \quad \boxed{y = 0}$$

So, we get tangnet parallel to y-axis at (1, 0).

(4) **Asymptotes :**

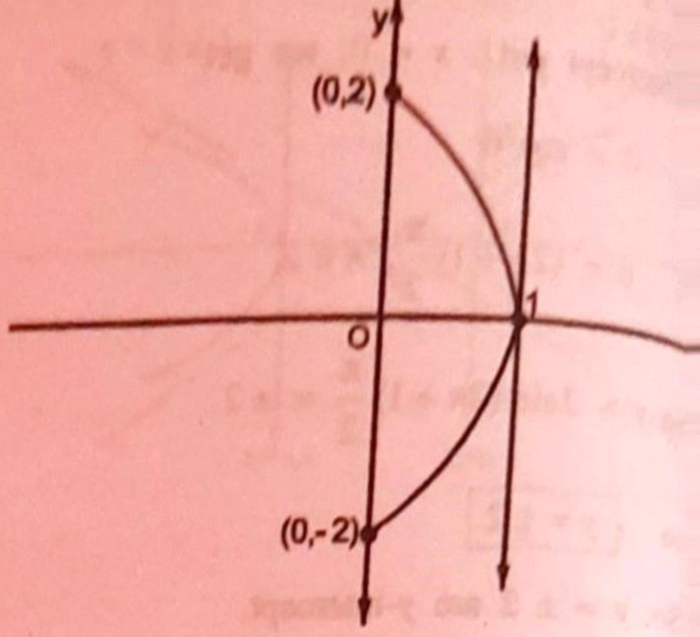
Asymptotes parallel to axes :

Here we can not find any limiting value of θ, for which one variable is finite and other is infinite. Therefore any asymptotes parallel to axes are not possible.

Oblique asymptotoes :

There does not axist any value of θ, for which both variable $x \to \infty$ and $y \to \infty$

$\therefore$ oblique asymptotes is not possible.



3. $x = 4t^2 - 4t, \quad y = 1 - 4t^2$:

(1) **Intercept :**

x-intercept : put y = 0, we get

$$0 = 1 - 4t^2$$

$$\therefore \quad 4t^2 = 1$$

$$\therefore \quad t^2 = \frac{1}{4}$$

$$\therefore \quad t = \pm\frac{1}{2}$$

$x = 4\left(\frac{1}{4}\right) - 4\left(\frac{1}{2}\right)$,	$x = 4\left(\frac{1}{4}\right) - 4\left(-\frac{1}{2}\right)$
$= 1 - 2$	$= 1 + 2$
$\therefore \quad \boxed{x = -1}$	$\therefore \quad \boxed{x = 3}$

So, $x = -1$ and x = 3 are x–intercepts.

y-intercept : Put $x = 0$, we get,

$$0 = 4t^2 - 4t$$

$$\therefore \quad 0 = 4t(t - 1)$$

$$\therefore \quad \boxed{t = 0} \quad \text{or} \quad \boxed{t = 1}$$

$y = 1 - 0$	$y = 1 - 4$
$\therefore \quad \boxed{y = 1}$	$\boxed{y = -3}$

So, $y = 1$ and $y = -3$ are y-intercepts.

(2) Extent : We know that

$(2t-1)^2 \geq 0$

$\Rightarrow 4t^2 - 4t + 1 \geq 0$

$\Rightarrow 4t^2 - 4t \geq -1$

$\Rightarrow \boxed{x \geq -1}$

$t^2 \geq 0$

$\Rightarrow -4t^2 \leq 0$

$\Rightarrow 1 - 4t^2 \leq 1$

$\Rightarrow \boxed{y \leq 1}$

(3) **Tangent parallel to axes :**

$$\frac{dy}{dx} = \frac{-8t}{8t-4} = \frac{-2t}{2t-1}$$

Tangent parallel to x-axis : We know that

$$\frac{dy}{dx} = 0 \text{ if } \frac{-2t}{2t-1} = 0,$$

if $-2t = 0$,

if $t = 0$

So, $x = 4t^2 - 4t$

$\therefore \boxed{x=0}$

$y = 1 - 4t^2$

$\therefore \boxed{y=1}$

So, we get tangent parallel to x-axis at (0, 1.).

Tangent parallel to y-axis : We know that

$$\frac{dy}{dx} \to \infty, \quad \text{if } \frac{-2t}{2t-1} \to \infty,$$

if $2t - 1 = 0$,

if $2t = 1$,

if $t = \frac{1}{2}$

So, $x = 4\left(\frac{1}{4}\right) - 4\left(\frac{1}{2}\right)$

$= 1 - 2$

$\therefore \boxed{x=-1}$

$y = 1 - 4\left(\frac{1}{4}\right)$

$= 1 - 1$

$\therefore \boxed{y=0}$

So, we get tangent parallel to y-axis at (– 1, 0).

(4) **Asymptotes :**

Asymptotes parallel to axies :

Here we can not find any limiting value of t, for which one variable is finite and other infinite. Therefore asymptotes parallel to axes are not possible.

Ex. 8 : Express the following in parametric form using the given substitution. Then sketc the curve.

1. $x^2 - 2xy + y^2 + y = 0,\ x - y = t$

Solution : $x^2 - 2xy + y^2 + y = 0$

$\Rightarrow\ (x-y)^2 + y = 0$

$\therefore\ t^2 + y = 0$

$\therefore\ \boxed{y = -t^2}$

Also, $x - y = t$

$\Rightarrow\ x + t^2 = t$

$\Rightarrow\ \boxed{x = t - t^2}$

(1) **Intercepts** :

x-intercept : put $y = 0$, we get $0 = t$

$\therefore\ \boxed{x = 0}$

So, $x = 0$ is x-intercept.

y-intercept : put $x = 0$, we get

$0 = t(1 - t)$

$\therefore\ t = 0$ or $t = 1$

$\therefore\ \boxed{y = 0}$ $\boxed{y = -1}$

So, $y = 0$ and $y = -1$ are y-intercepts.

(2) **Extent :** We know that

$$\left(t-\frac{1}{2}\right)^2 \geq 0$$

$$\therefore \quad -\left(t-\frac{1}{2}\right)^2 \leq 0$$

$$\Rightarrow \quad -\left(t^2-t+\frac{1}{4}\right) \leq 0$$

$$\Rightarrow \quad -t^2+t-\frac{1}{4} \leq 0$$

$$\therefore \quad t-t^2 \leq \frac{1}{4}$$

$$\therefore \quad \boxed{x \leq \frac{1}{4}}$$

$$t^2 \geq 0$$

$$\Rightarrow \quad -t^2 \leq 0$$

$$\therefore \quad \boxed{y \leq 0}$$

(3) **Tangent parallel to axes :**

$$\frac{dy}{dx} = \frac{-2t}{1-2t}$$

Tangent parallel to x-axis : We knowt hat

$$\frac{dy}{dx} = 0, \text{ if } 2t = 0 \text{ if } t = 0$$

So, $\boxed{x=0}$ and $\boxed{y=0}$

So, tangent parallel to x-axis at (0, 0)

Tangent parallel to y-axis : We know that

$$\frac{dy}{dx} \to \infty \text{ if } \frac{-2t}{1-2t} \to \infty,$$

if $1 - 2t = 0$,

if $2t = 1$,

if $t = \frac{1}{2}$,

So, $x = \frac{1}{2} - \frac{1}{4}$

$$\therefore \quad \boxed{x = \frac{1}{4}}$$

$$\boxed{y = -\frac{1}{4}}$$

So, we get tangent parallel to y-axis at $\left(\frac{1}{4}, -\frac{1}{4}\right)$.

Asymptotes :

mptotes parallel to axes :

Here we can not find any limiting value of t, for which one variable is finite and other infinite. Therefore asymptotes parallel to axes are not possible.

lique asymptotes :

Here if $x \to \infty$, then $y \to \infty$ and $y \to \infty$.

So, there is possibility of oblique asymptotes which is given by

$$y = mx + c$$

where,

$$m = \lim_{t \to \infty} \left(\frac{dy}{dx}\right)$$

$$= \lim_{t \to \infty} \left(\frac{-2t}{1-2t}\right)$$

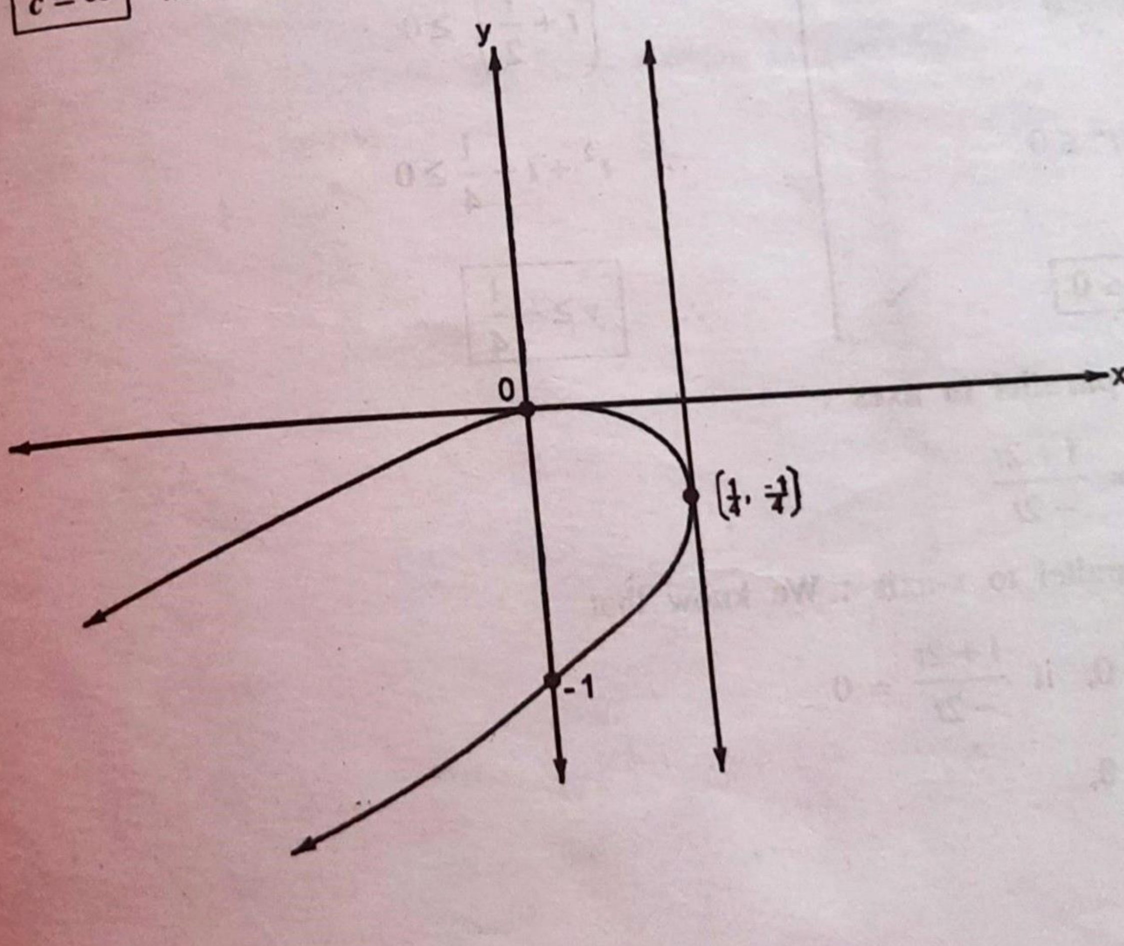
$$= \lim_{t \to \infty} \left(\frac{-2t}{t(1/t - 2)}\right)$$

$\therefore$ $$\boxed{m = 1}$$

Also $$c = \lim_{t \to \infty} (y - mx) = \lim_{t \to \infty} (y - x)$$

$$= \lim_{t \to \infty} (-t^2 - t + t^2)$$

$\boxed{c = \infty}$ which is not possible.



- **Theorem-2 :**

Obtain parametric equation of Cycloid.

OR Prove that a cycloid obtain by rolling a circle of radius 'a' is given by

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$$

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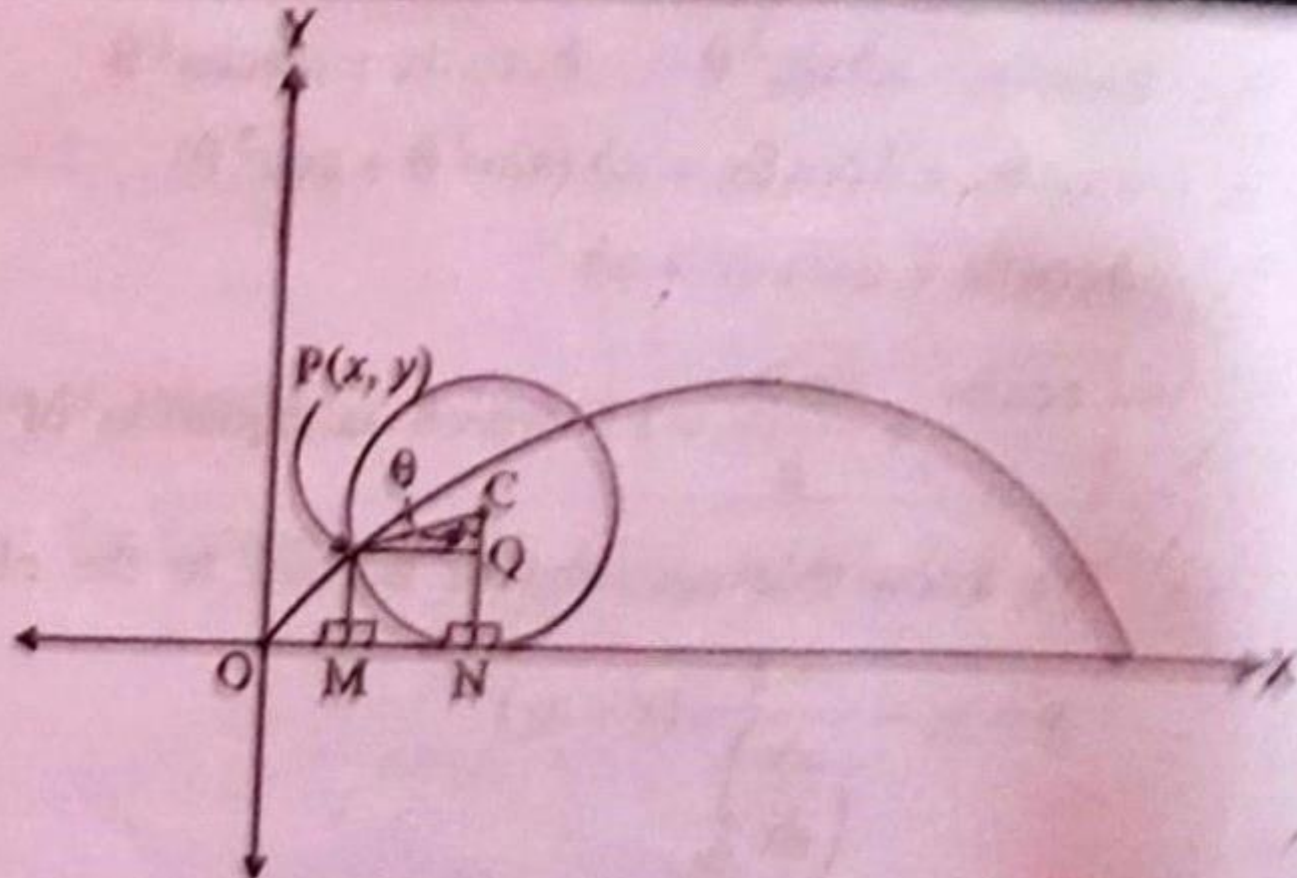
Proof :

If a circle with centre C and radius 'a' rolls along X-axis as shown in fig.



At starting, fixed point P is at origin 0. Let P(x, y) be any point on cycloid.

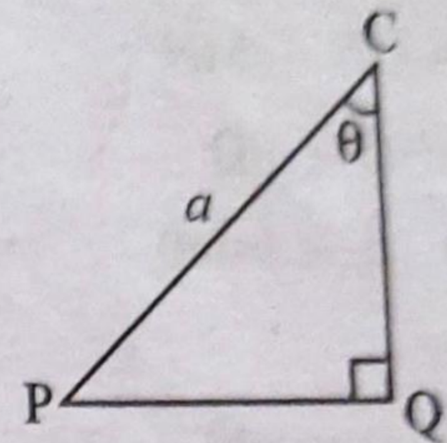
Rotate radius CP through angle θ as shown in fig.

Draw $\overline{CN}$ perpendicular to X-axis, $\overline{PM} \perp$ X-axis and $\overline{PQ} \perp \overline{CN}$, $CN = PC = a$.

We know that arc length $PN = a\theta$ $\quad$ ($\because$ arc length = radius X angle.)

Also from right angle ΔPCQ,

$$\cos\theta = \frac{CQ}{PC} = \frac{CQ}{a}$$



$$\therefore \boxed{CQ = a\cos\theta}$$

and $\sin\theta = \dfrac{PQ}{a}$ $\quad \therefore \quad \boxed{PQ = a\sin\theta}$

Also from fig. ON = arc length $PN = a\theta$

From fig.

$$\begin{aligned} x &= OM \\ &= ON - MN \\ &= a\theta - PQ \\ &= a\theta - a\sin\theta \end{aligned}$$

$$\therefore \boxed{x = a(\theta - \sin\theta)}$$

Also,

$$\begin{aligned} y &= PM \\ &= CN - CQ \\ &= a - a\cos\theta \end{aligned}$$

$$\boxed{y = a(1 - \cos\theta)}$$

Hence, $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ are required parametric equation of cycloid.