Unit **2**

SPECIAL THEORY OF RELATIVITY

2.1 INTRODUCTION :

According to classical theories at the beginning of the twentieth century, the measurement of space, time and mass is absolute. The theory of relativity was first formulated by Albert Einstein in the year 1905. He showed that measurement of space or time in the universe is not absolute but relative. The measurements depend upon the state of motion of the observer as well as upon the quantities that are being measured.

The theory which deals with the relativity of motion and rest is called the theory of relativity. The theory of relativity is divided in to two parts: special theory and general theory. The special theory of relativity deals with objects and systems which are either moving at a constant speed with respect to one another or are at rest. The general theory of relativity deals with objects or systems which are speeding up or slowing down or simply accelerating with respect to one another.

(i) **Event :** An event is something that happens at a particular point in space and at a particular instant of time, independent of the reference frame.

(ii) **Observer :** An observer is a person or an equipment mean to observe and to take measurement about the event,

2.2 FRAME OF REFERENCE :

A body in motion can be located with reference to some coordinate system. This coordinate system is called the frame of reference. It is denoted as S : [O-XYZ].

If the coordinates of all the points of a body do not change with respect to time and frame of reference, the body is said to be at rest. But if the coordinates of all the points of a body change with respect to time and frame of reference, the body is said to be in motion.

The frame of reference is selected in such a way that the laws of nature may become fundamentally simpler in the frame of reference.

To understand the selection of frame of reference, consider two frames of reference S [O-XYZ] and S'[O'-X'Y'Z'] as shown in Fig. 2.1. Suppose that two observers sitting at O and O' in frames S and S^1 observe motion of the particle respectively. If O and O' are at rest, they will observe the same motion of the particle. But if they are in relative

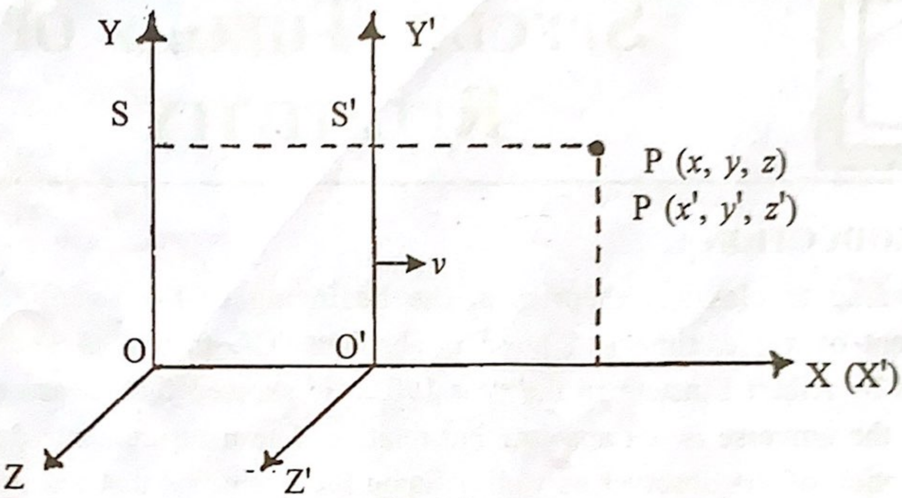


FIG. 2.1

motion with respect to each other, their observations for the same motion of a particle are different.

To understand we take two examples:

(a) Suppose observer A is on earth and B is on the sun and they observe motion of the moon. The moon appears to move in a circular path to A, while it appears to move on a wavy path to B.

(b) If a person sitting in a train moving with a constant velocity drops a stone from the window. He observes that the path of the falling stone is straight line. But for a person standing near the track, stone appears to move in a parabolic path.

There are two types of frames of reference :

(i) Inertial or unaccelerated frames of reference

(ii) Non-inertial or accelerated frames of reference

(i) Inertial frames :

The frame of reference in which law of inertia holds true is called an inertia! frame of reference. In other words a reference frame is said to be inertial when bodies in this frame obey Newton's law of inertia as well as other laws of Newtonian mechanics. In this frame, a body is at rest or moves with a constant velocity. An observer in this frame is called an inertial observer.

(ii) Non-inertial frames :

The frame of reference in which law of inertia does not hold true is called non-inertial frame of reference. In other words an accelerated frame of reference is said to by non-inertial frame (when a body not acted upon by any external force). In this frame, the

Newton's laws are not valid. For example, a frame of reference fixed on the earth is not an inertial frame since the earth is rotating about its own axis as well as revolving around the sun.

Generally a frame of reference fixed on a pole star may be considered an absolute frame where we can observe absolute rest and any motion measured with respect to this frame is called an absolute motion.

2.3 GALILEAN TRANSFORMATIONS :

The Galilean transformations are used to transform the coordinates of position and time from one inertial frame to the other.

Consider two inertial frames of reference S [O - XYZ] and S' [O' - X'Y'Z'], in which S is stationary, while S' is moving with velocity v in positive X direction. Initially (at time t = 0), their origins O and O' coincide with each other. Assume that some event occurs at a point P (as shown in Fig. 2.2) and observer sitting at points O and O' and record the space coordinates and time in their frame. An observer O measures coordinates x, y, z and time interval t for the event at P in frame S, while observer O' measures coordinates x', y', z' and time interval t' for the same event at P in S^1 frame. Considering the concept of absolute nature of time, time remains same in both the frames. The relations or transformation equations between the coordinates are written as

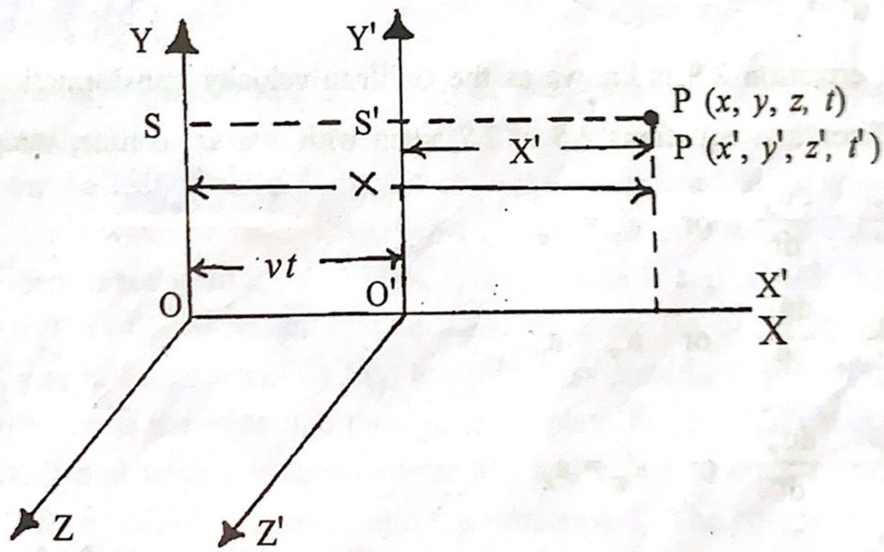


FIG. 2.2

$$x' = x - vt \qquad \text{... ... ... (2.1)}$$

$$y' = y \qquad \text{... ... ... (2.2)}$$

$$z' = z \qquad \text{... ... ... (2.3)}$$

$$t' = t \qquad \text{... ... ... (2.4)}$$

The set of these equations 2.1 to 2.4 are called Galilean transformations.

To derive velocity transformations differentiate equations 2.1 to 2.3 with respect to time, $t' = t$ or $\frac{d}{dt'} = \frac{d}{dt}$

$$\frac{dx'}{dt} = \frac{dx}{dt} - v \text{ or } u_x' = u_x - v \qquad \text{... ... ... (2.5)}$$

$$\frac{dy'}{dt'} = \frac{dy}{dt} \text{ or } u_y' = u_y \qquad \text{... ... ... (2.6)}$$

$$\frac{dz'}{dt'} = \frac{dz}{dt} \text{ or } u_z' = u_z \qquad \text{... ... ... (2.7)}$$

Multiplying equations 2.5, 2.6 and 2.7 with unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively, and adding the results we get...

$$\hat{i}u_x' + \hat{j}u_y' + \hat{k}u_z' = \hat{i}u_x +, \hat{j}u_y + \hat{k}u_z - v$$

$$\text{or } \vec{u}' = \vec{u} - \vec{v} \qquad \text{... ... ... (2.8)}$$

The equation 2.8 is known as the Galilean velocity transformations.

Differentiate equations 2.5 to 2.7 again with respect to time, we get

$$\frac{du_{x'}}{dt'} = \frac{du_x}{dt} \text{ or } a'_x = a_x \qquad \text{... ... ... (2.9)}$$

$$\frac{du_{y'}}{dt'} = \frac{du_y}{dt} \text{ or } a'_y = a_y \qquad \text{... ... ... (2.10)}$$

$$\frac{du_{z'}}{dt'} = \frac{du_z}{dt} \text{ or } a'_z = a_z \qquad \text{... ... ... (2.11)}$$

Multiplying equations 2.9, 2.10 and 2.11 with unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ respectively and adding we get

$$\therefore \vec{a}' = \vec{a} \qquad \text{... ... ... (2.12)}$$

Thus, the acceleration of a particle is observed same in all inertial frames of references moving relative to each other with constant velocity, i.e. **the acceleration is invariant under Galilean transformations.**

Luminiferous Ether :

According to the electromagnetic wave theory, light waves are transverse wave and can be polarized. Since longitudinal waves of sound require medium for propagation, it was assumed that transverse waves of light also requires medium for propagation, called ether. Ether is purely hypothetical medium assume to having following features;

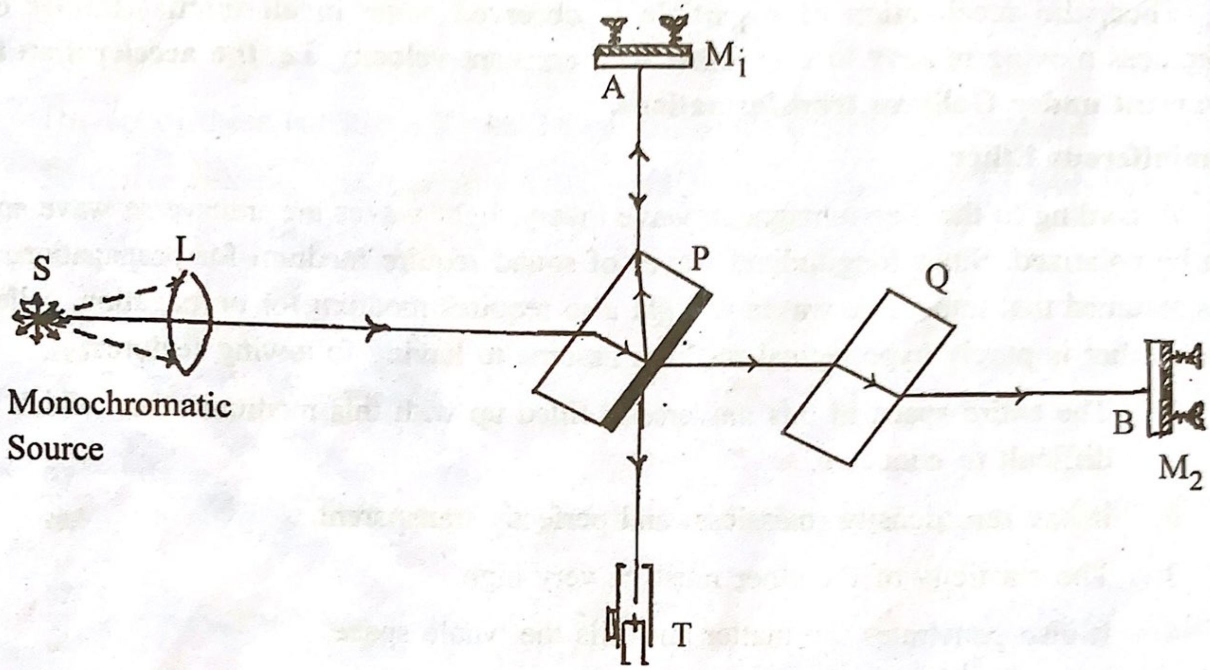
1. The entire space in this universe is filled up with this medium ether, which is difficult to conceive.
2. It has zero density (massless) and perfectly transparent.
3. The elasticity of the ether must be very high.
4. It also penetrates the matter and fills the whole space.

It was assumed that the earth moves through ether without producing any disturbance. So if the ether hypothesis is correct then it is possible to determine the absolute velocity of the earth ith respect to ether frame (which is considered to be at rest). To detect the luminiferous ether, Michelson and Morley experiment was performed.

2.4 MICHELSON - MORLEY EXPERIMENT :

Construction & Working :

The experimental arrangement of Michelson's interferometer is shown in Fig. 2.3. A monochromatic light beam emitted from source S, become parallel after passing through lens L is incident on the half-silvered glass plate P which is inclined at an angle 45° to the incident beam. Each wave of the incident beam is split in to two waves having same amplitude. One of them is reflected by glass plate P, in direction A and other is transmitted through glass plate P, in the direction B. The reflected wave (in direction A) travels towards mirror M_1 and reflected back (by M_1) towards glass plate P. A part of this wave is then transmitted towards the telescope through glass plate P. The other part of wave refracted (transmitted) and travels towards mirror M_2 and reflected back to plate P. A part of this wave is then partially reflected into the telescope T. The compensating plate Q is placed in the path of the transmitted beam so as to make the path of the two beams equal in glass medium. When both the waves A and B enter in the telescope, they interfere with each other and produced the interference fringes formed can be viewed by telescope T. Both the mirrors M_1 and M_2 kept at the same distance D from glass plate P.



Here, S : Monochromatic source of light (Generally sodium lamp for yellow light)

L : Lens P : Half silvered plate Q : Compensating plate

M_1 and M_2 : Plane mirrors T : Telescope

FIG. 2.3

If the earth and hence the apparatus is at rest in ether, the two waves take equal time to return the glass plate P and hence meet in the same phase at glass plate and also

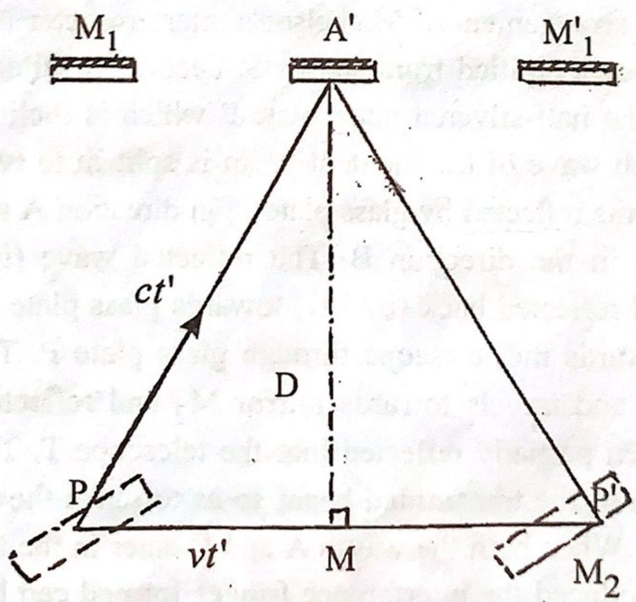


FIG. 2.4

in the telescope. But actually the whole apparatus is moving along with the earth with velocity v in the ether. Suppose that the direction of motion of earth is in the direction of the initial beam (i.e. from left to right direction) as shown in the Fig. 2.4. Due to the motion of the earth, the optical paths traversed by both the beams are not the same and time taken by two waves to travel to the mrrors and back to P will be different in case.

→ **In direction B i.e. Path P – M_2 – P :**

The transmitted beam travels towards M_2 with velocity $c - v$, after reflection at mirror M_2, it travels towards plate P with velocity $c + v$. Hence the time required by this beam in the round trip journey is given by

$$t_2 = \frac{D}{c - v} + \frac{D}{c + v} = \frac{2Dc}{c^2 - v^2} \quad \text{... ... ... (2.13)}$$

$$\therefore \quad t_2 = \frac{2Dc}{c^2\left[1 - \frac{v^2}{c^2}\right]} = \frac{2D}{c}\left[1 - \frac{v^2}{c^2}\right]^{-1} \quad \text{... ... ... (2.14)}$$

using binomial theorem,

$$\therefore \quad t_2 \approx \frac{2D}{c}\left[1 + \frac{v^2}{c^2}\right] \quad \text{... ... ... (2.15)}$$

→ **In direction A i.e. path P - A' - P' :**

Due to the motion of the earth and apparatus, mirror M_1 will be shifted to A' in time t' and the path taken by the beam in direction A is P-A'-P' (in place of P-M_1-P) as shown in Fig. 2.4.

The distance travelled by the light beam is PA' = ct', the distance traveled by the apparatus (in the same time t' and in the right hand side) is vt' and MA' = D, for triangle PA'M we have :

$$PA'^2 = PM^2 + MA'^2 \quad \text{... ... ... (2.16)}$$

but PA' = ct', PM = vt' and MA' = D then,

$$\therefore \quad c^2 t'^2 = v^2 t'^2 + D^2 \quad \text{... ... ... (2.17)}$$

$$\text{or} \quad c^2 t'^2 - v^2 t'^2 = D^2$$

$$\therefore \quad (c^2 - v^2)\, t'^2 = D^2 \quad \text{... ... ... (2.18)}$$

$$\therefore \ t'^2 = \frac{D^2}{\left(c^2 - v^2\right)}$$

$$\therefore \ t'^2 = \frac{D^2}{c^2\left(1 - \frac{v^2}{c^2}\right)} \quad \text{... ... ... (2.19)}$$

$$\therefore \ t' = \frac{D}{c\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}}$$

$$\text{or} \quad t' = \frac{D}{c}\left[1 - \frac{v^2}{c^2}\right]^{-\frac{1}{2}} \quad \text{... ... ... (2.20)}$$

using binomial theorem,

$$\therefore \ t' \approx \frac{D}{c}\left[1 + \frac{v^2}{2c^2}\right] \quad \text{... ... ... (2.21)}$$

If t_1 is the time taken by the beam in round trip journey P-A'-P', then

$$t_1 = 2t' = \frac{2D}{c}\left[1 + \frac{v^2}{2c^2}\right] \quad \text{... ... ... (2.22)}$$

Hence, the difference in the time taken by the two light beams for their round trips is given by

$$T = t_2 - t_1 = \frac{2D}{c}\left[1 + \frac{v^2}{c^2}\right] - \frac{2D}{c}\left[1 + \frac{v^2}{2c^2}\right] \quad \text{... ... ... (2.23)}$$

$$= \frac{2D}{c}\left[\frac{v^2}{2c^2}\right]$$

$$\therefore \quad T = \frac{Dv^2}{c^3} \quad \text{... ... ... (2.24)}$$

A time difference of one period (T) gives a path difference of one wave length (λ); Corresponds to a displacement of one fringe across a particular point (i.e. crosswire of telescope) in the field view of telescope,

Therefore, a time period $T = t_2 - t_1$ causes a path difference $x = n\lambda$,

(As velocity $= \frac{\text{distance}}{\text{time}}$ or $c = \frac{\lambda}{T}$ or $\lambda = cT$)

$$\therefore \quad n\lambda = cT = c\left(\frac{Dv^2}{c^3}\right) = \frac{Dv^2}{c^2} \quad \text{... ... ... (2.25)}$$

and Displacement of fringes, $n = \frac{Dv^2}{\lambda c^2}$ ... ... (2.26)

The displacement of the fringes cannot be noticed since apparatus is at rest with respect to the observer. Therefore apparatus is slowly turned through 90° so that two beams interchange their paths and

$$\text{Displacement of fringes} = -\frac{Dv^2}{\lambda c^2}$$

Thus, the total displacement of fringes now becomes

$$n = \frac{Dv^2}{\lambda c^2} - \left(-\frac{Dv^2}{\lambda c^2}\right)$$

$$n = \frac{2Dv^2}{\lambda c^2} \quad \text{... ... ... (2.27)}$$

This is the necessary formula for displacement of fringes

Example 2.1 : In Michelson - Morley experiment, the wave length of the light is 5890 Å, the distance for the round trip journey is 11 m, the orbital velocity of the earth is 3×10^4 ms^{-1} and velocity of light is 3×10^8 ms^{-1} calculate the displacement of the fringes.

Given,

- Wavelength $\lambda = 5890$ Å $= 5.89 \times 10^{-7}$ m
- distance for the round trip, D = 11 m
- velocity of the earth, $v = 3 \times 10^4$ ms^{-1}
- velocity of light, $c = 3 \times 10^8$ ms^{-1}

Formula,

Total path difference,

$$n = \frac{2Dv^2}{\lambda c^2}$$

$$= \frac{2 \times 11 \times \left[3 \times 10^4\right]^2}{5.89 \times 10^{-7} \times \left[3 \times 10^8\right]^2}$$

$$= 0.3735$$

$$\approx 0.37$$

The theoretical value of the path difference is 0.37 of a fringe. Thus, 0.37 fringe will be displaced across the cross wire of telescope.

2.5 THE NEGATIVE RESULTS OF MTCHELSON - MORLEY EXPERIMENT :

The theoretical value of the path difference is 0.37. But no such displacement is observed experimentally.

If we try lo analyze this negative result then from equation 2.27 we can say that the displacement of fringes: n becomes zero if :

(i) The earth is stationary, i.e. $v = 0$. But we know that earth is not stationary it has orbital velocity: 3×10^4 m/s. So the assumption that the earth is stationary is not true.

(ii) It may be possible that for some time the relative motion between earth and ether becomes zero, i.e. both are moving in the same direction. But we know that the earth completes its one rotation in a year and in such elliptical (or for simplicity circular) motion the ether and earth are not moving in the same direction throughout the year and the relative motion between earth and ether never becomes zero throughout the year. Hence the experiment was performed in different seasons of the year and at different places. But no such displacement is observed.

(iii) The velocity of light is infinite, which is also not possible. We know that the speed of the light is $c = 3 \times 10^8$ m/s.

(iv) The arm length of the instrument is zero (i.e. D = 0) but we know that D has some finite value (nearly 10 m).

By considering above facts finally we come to a conclusion that the assumption we have made about the existence of medium ether in the universe is not true. In reality no such medium exists or in justifiable words if ether exist, it is undetectable.

The negative results of the Michelson - Morley experiment led to the following conclusions :

(1) If ether exists, is undetectable,

(2) The speed of light is invariant in free space and remains constant c in all directions. It is independent of the motion of the source or the observer.

2.6 EINSTEIN THEORY OF RELATIVITY :

Upon examining a large number of problems of the detection of ether (or ether wind) and the experiments which had been performed, in 1905 Einstein draw two very important conclusions. These are known as the fundamental postulates of the special theory of relativity. These postulates are :

(i) The laws of physics are same in all inertial frames of references which are moving with a constant velocity relative to each other.

According to this postulate it is impossible to demonstrate "absolute motion". Hence this postulate is also known as the principle of relativity.

(ii) The speed of light in free space has the same value c in every inertial frame.

According to this postulate the speed of light is the same in all directions, no matter whether the source is moving or stationary or the observer. Therefore this postulate is , own as the principle of constancy of the speed of light.

The theory based on these two postulates and applies to all inertial frames is called the special theory of relativity.

The theory, which deal with the accelerated systems is called the general theory of relativity.

2.7 LORENTZ TRANSFORMATION OF SPACE AND TIME :

Consider two inertial frames of reference S [O-XYZ] and S' [O'-X'Y'Z'], in which S is stationary and S' is moving with constant velocity v in positive X direction. Initially (at time t = 0) when the origins O and O' coincide with each other, a light pulse is generated at the origin. When this pulse reaches a point P, two observers sitting at O and O' record the space coordinates and time interval for the event in their frames as (x, y, z. t) and (x', y', z', t') respectively.

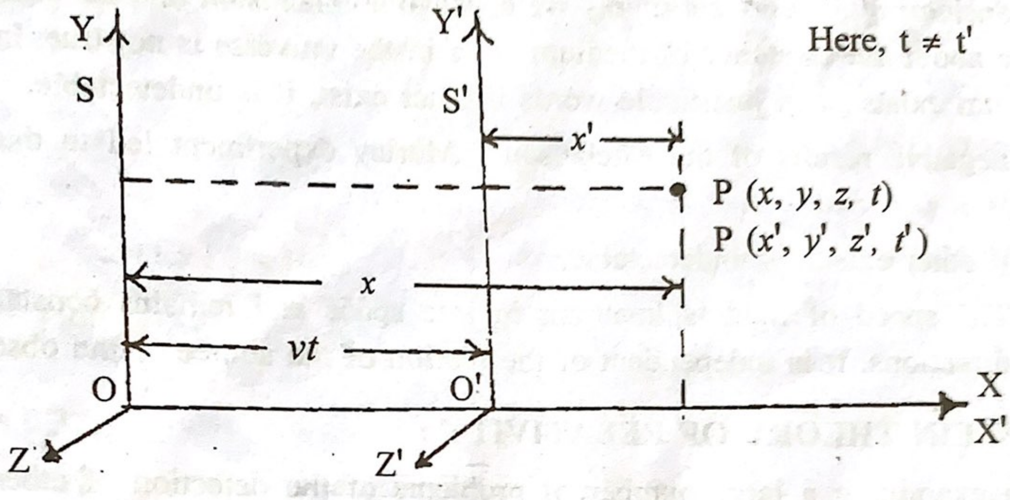


FIG. 2.5

We know that velocity $= \dfrac{\text{distance}}{\text{time}}$

When the pulse is observed from S, the distance $= (x^2 + y^2 + z^2)^{\frac{1}{2}}$

and $c = \dfrac{\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}}{t}$

or

$$x^2 + y^2 + z^2 = c^2t^2 \text{ or } x^2 + y^2 + z^2 - c^2t'^2 = 0 \quad \ldots \quad \ldots \quad \ldots (2.28)$$

Similarly for observer O' in system S' is

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \text{ or } x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \quad \ldots \quad \ldots \quad \ldots (2.29)$$

As the velocity of S' is only in positive X direction,

$$y = y' \text{ and } z = z' \quad \ldots \quad \ldots \quad \ldots (2.30)$$

Comparing equations (2.28) and (2.29) and using equation (2.30) we have

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \quad \ldots\ldots\ldots \quad \ldots \quad \ldots (2.31)$$

Now for the transformation equation relating to x and x', let us put

$$x' = \lambda\,(x - vt) \quad \ldots \quad \ldots \quad \ldots (2.32)$$

λ being independent of x and t.

Since motion is relative, we may assume that S is moving relative to S' with velocity $-v$ along the positive x direction.

$$\therefore \quad x = \lambda'\,(x' + vt') \quad \ldots \quad \ldots \quad \ldots (2.33)$$

Now, substituting the value of x' from equation (2.32) in equation (2.33)

$$x = \lambda' [\lambda (x - vt) + vt']$$

$$\therefore \quad \frac{x}{\lambda'} = \lambda x - \lambda vt + vt'$$

$$\therefore \quad vt' = \frac{x}{\lambda'} + \lambda vt - \lambda x$$

$$= \lambda \left[\frac{x}{\lambda\lambda'} + vt - x\right]$$

$$\therefore \quad t' = \lambda \left[t + \frac{x}{v\lambda\lambda'} - \frac{x}{v}\right]$$

$$\therefore \quad t' = \lambda \left[t - \frac{x}{v}\left(1 - \frac{1}{\lambda\lambda'}\right)\right] \qquad \text{... ... ... (2.34)}$$

Now substituting the value of x' from equation (2.32) and value of t' from equation (2.34) in equation (2.31), we have

$$x^2 - c^2 t^2 = \lambda^2 (x - vt)^2 - c^2 \lambda^2 \left[t - \frac{x}{v}\left(1 - \frac{1}{\lambda\lambda'}\right)\right]^2$$

$$x^2 - c^2 t^2 - \lambda^2 (x^2 - 2vxt + v^2 t^2) + c^2 \lambda^2 \left[t - \frac{x}{v}\left(1 - \frac{1}{\lambda\lambda'}\right)\right]^2 = 0$$

$$\therefore \quad x^2 - c^2 t^2 - \lambda^2 (x^2 - 2vxt + v^2 t^2) +$$

$$c^2\lambda^2 \left[t^2 - \frac{2xt}{v}\left(1 - \frac{1}{\lambda\lambda'}\right) + \frac{x^2}{v^2}\left(1 - \frac{1}{\lambda\lambda'}\right)^2\right] = 0$$

$$x^2 \left[1 - \lambda^2 + \frac{c^2\lambda^2}{v^2}\left(1 - \frac{1}{\lambda\lambda'}\right)^2\right] + xt \left[2\lambda^2 v + c^2\lambda^2 \left\{-\frac{2}{v}\left(1 - \frac{1}{\lambda\lambda'}\right)\right\}\right]$$

$$+ t^2 [-c^2 - \lambda^2 v^2 + c^2 t^2] = 0 \qquad \text{... ... ... (2.36)}$$

Since the equation is an identify; the coefficients of various powers of x and t vanish separately.

For coefficients of x^2 :

$$1 - \lambda^2 + \frac{c^2\lambda^2}{v^2}\left(1 - \frac{1}{\lambda\lambda'}\right)^2 = 0$$

$$\therefore \quad 1 - \lambda^2 + \frac{c^2\lambda^2}{v^2}\left(1 - \frac{2}{\lambda\lambda'} + \frac{1}{\lambda^2\lambda'^2}\right) = 0$$

$$\therefore \quad 1 - \lambda^2 + \frac{c^2}{v^2}\left(\lambda^2 - \frac{2\lambda}{\lambda'} + \frac{1}{\lambda'^2}\right) = 0 \qquad \ldots \ldots \ldots (2.36)$$

For coefficient of xt :

$$2\lambda^2 v + c^2\lambda^2\left\{-\frac{2}{v}\left(1 - \frac{1}{\lambda\lambda'}\right)\right\} = 0$$

$$\therefore \quad 2\lambda^2 v - \frac{2c^2\lambda^2}{v}\left(1 - \frac{1}{\lambda\lambda'}\right) = 0$$

$$\therefore \quad \lambda^2 v^2 - c^2\lambda^2\left(1 - \frac{1}{\lambda\lambda'}\right) = 0$$

$$\therefore \quad v^2 - c^2\left(1 - \frac{1}{\lambda\lambda'}\right) = 0$$

$$\therefore \quad v^2 - c^2 + \frac{c^2}{\lambda\lambda'} = 0$$

$$\therefore \quad (v^2 - c^2)\,\lambda\lambda' + c^2 = 0 \qquad \ldots \ldots \ldots (2.37)$$

For coefficient of t^2 :

$$-c^2 - \lambda^2 v^2 + c^2\lambda^2 = 0$$

$$\therefore \quad c^2 + \lambda^2 v^2 - c^2\lambda^2 = 0$$

$$\therefore \quad (v^2 - c^2)\,\lambda^2 + c^2 = 0 \qquad \ldots \ldots \ldots (2.38)$$

Comparing equations (2.37) and (2.38), we get

$$\lambda = \lambda' \qquad \ldots \ldots \ldots (2.39)$$

From equation (2.38) we get

$$\lambda^2 = -\frac{c^2}{v^2 - c^2} = \frac{c^2}{c^2 - v^2} \qquad \ldots \ldots \ldots (2.40)$$

$$\therefore \quad \lambda^2 = \frac{1}{1 - \frac{v^2}{\ldots}}$$

$$\therefore\ \lambda = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{... ... ... (2.41)}$$

From equation (2.34), we have

$$t' = \lambda\left[t - \frac{x}{v}\left(1 - \frac{1}{\lambda^2}\right)\right] \qquad (\because \lambda = \lambda')$$

$$= \lambda\left[t - \frac{x}{v}\left(1 - \frac{c^2 - v^2}{c^2}\right)\right] \qquad \left[\because \text{ from equ. (2.41) } \lambda^2 = \frac{c^2}{c^2 - v^2}\right]$$

$$\therefore\quad t' = \lambda\left[t - \frac{x}{v}\left(\frac{c^2 - c^2 + v^2}{c^2}\right)\right]$$

$$= \lambda\left[t - \frac{x}{v}\frac{v^2}{c^2}\right]$$

$$\therefore\quad t' = \lambda\left[t - \frac{xv}{c^2}\right] \qquad \text{... ... ... (2.42)}$$

Substituting value of λ from equation (2.41) in equations (2.32) and (2.42) we get,

$$\left.\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\ t' &= \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}\right\} \qquad \text{... ... ... (2.43)}$$

Thus combining (2.30) and (2.43)

We have,

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{... ... ... (2.44)}$$

$$y' = y \qquad \text{... ... ... (2.45)}$$

$$z' = z \qquad \text{... ... ... (2.46)}$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{... ... ... (2.47)}$$

These are called **Lorentz - transformations equations.**

If we use standard notations putting

$$\beta = \frac{v}{c} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} \qquad \text{... ... ... (2.48)}$$

Lorentz transformations takes the form,

$$\left.\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - \beta^2}} = \gamma\,(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \beta^2}} = \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}\right\} \qquad \text{... ... ... (2.49)}$$

The inverse Lorents transformations can now be written as

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{... ... ... (2.50)}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Thus, the measurements of position and time are found to depend upon the frame of reference of the observer.

For very small velocities compare to velocity of light : V < < C then $\frac{v}{c}$, $1-\frac{v^2}{c^2}$, $\frac{v}{c^2}$ become negligible therefore equation 2.49 becomes

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

Which are Galilean transformations.

The Lorentz equations reduce to the Galilean transformations when relative velocity v is very small in comparison with velocity c of light.

2.8 LORENTZ - FITZGERALD LENGTH CONTRACTION :

Consider two inertial frames of reference S [O – XYZ] and S' [O' – X'Y'Z'], in which S is stationary and S' is moving with velocity v in positive X direction. Initially at t = 0, O and O' coincide with each other.

Now, assume that a rod of length L_0 is placed along the axis X' in the frame S' which is moving with velocity v with respect to frame S. Now the rod is at rest with respect to the observer O' (as shown in Fig. 2.6).

The original length of the rod L_0 is measured by O' in frame S' is given by

$$L_0 = x'_2 - x'_1 \quad \text{... ... ... (2.51)}$$

Where, x'_1 and x'_2 are the extrimities of the rod. If L is the length of the rod in frame S relative to which the rod is in motion with velocity v, from the Lorentz transformation equation 2.49 We get,

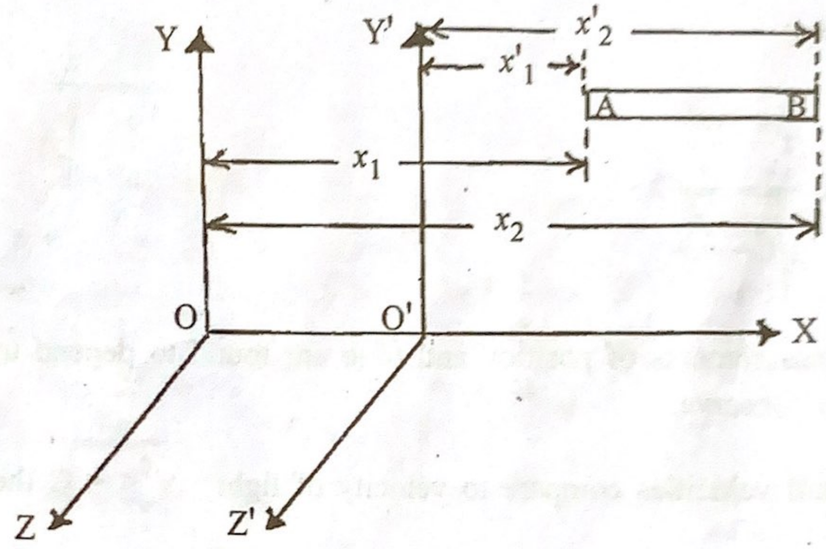


FIG. 2.6

$$x'_1 = \frac{x_1 - vt_1}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{... ... ... (2.52)}$$

and $$x'_2 = \frac{x_2 - vt_2}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{... ... ... (2.53)}$$

Hence,

$$L_0 = x'_2 - x'_1$$

$$= \frac{x_2 - vt_2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{x_1 - vt_1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{x_2 - x_1 - v(t_2 - t_1)}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{... ... ... (2.54)}$$

But $x_2 - x_1 = L$, the length of the rod measured in frame S, and the measurements should be simultaneous in frame S, therefore $t_2 = t_1$

Thus,

$$L_0 = \frac{x_2 - x_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \qquad L_0 = \frac{L}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \ldots \ldots \ldots (2.55)$$

$$\text{or} \qquad L = L_0\sqrt{1-\frac{v^2}{c^2}} \qquad \ldots \ldots \ldots (2.56)$$

Equation 2.56 shows, the length of the rod in motion with respect to an observer appears to be shorter than when it is at rest with respect to him. The phenomenon is known as the Lorentz-Fitzgerald length contraction. It should be noted that the effects of length contraction become significant only when the velocity of the objects approaches the velocity of light. Thus if v = 0.9 c, ratio $\frac{L}{L_0}$ is about 0.44, the following cases will clear this explanation :

1. If v = 0.1 c then $L = 0.99499\ L_0$
2. If v = 0.5 c then $L = 0.86603\ L_0$
3. If v = 0.9 c then $L = 0.43589\ L_0$
4. If v = c then L = 0

Last case is not possible, that means for heavy object (massive body) can not travel with the velocity of light, in another words, the maximum limit of the velocity of any massive body is c.

Example 2.2 : A rocket ship is 100 m long on the ground. When it is in flight, its length is 99 m to an observer on the ground. What is its speed ?

Solution :

We know that

$$L = L_0\sqrt{1-\frac{v^2}{c^2}} \qquad L_0 = 100 \text{ m}, L = 99 \text{ m}$$

$$\therefore \qquad 99 = 100\sqrt{1-\frac{v^2}{c^2}}$$

$$\therefore \qquad \frac{99}{100} = \sqrt{1-\frac{v^2}{c^2}} \text{ or } (0.99)^2 = 1-\frac{v^2}{c^2}$$

$$\therefore \qquad \frac{v^2}{c^2} = 1-(0.99)^2$$

$\therefore \quad v^2 = (1 - 0.9801)\, c^2$

$= 0.0199\, c^2$

$= 4.23 \times 10^7$ m/s

$\therefore \quad v = 4.23 \times 10^9$ cm/s

Example 2.3 : A rod has length 1 m. When the rod is in a satellite moving with a velocity that is half of the velocity of light relative to laboratory, what is the length of the rod as determined by an observer (a) in the satellite and (b) in the laboratory.

Solution :

(a) The observer in the satellite is at rest relative to rod hence the length measured by him is 1 m.

(b) The length 'L' of the rod in the laboratory is given by

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

$$= 1\sqrt{1 - \left(\frac{0.5\,c}{c}\right)^2}$$

$$= 1 \times (0.75)^{1/2} = 0.866 \text{ m} \quad \text{or} \quad 86.6 \text{ cm}$$

$\therefore L = 86.6$ cm

2.9 TIME DILATION :

Consider two inertial frames of reference S [O - XYZ] and S' [O' - X' Y' Z'], in which S is stationary while S' is moving with velocity v in positive *X*-direction. At time t = 0; O and O' coincide with each other. Further consider a clock situated at position x' in moving frame S'. Suppose that t'_1 and t'_2 are starting time and end time of any event respectively, and time recorded by the observer in frame S' at any two instants.

The time interval of that event measured by him is given by

$$t_0 = t'_2 - t'_1 \qquad \dots \dots \dots (2.57)$$

The observer in frame S measures these instants as

$$t_1 = \frac{t'_1 + \frac{v'x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \dots \dots \dots (2.58(a))$$

and $$t_2 = \frac{t'_2 + \frac{v'x}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{... ... ... (2.58(b))}$$

respectively.

∵ Using inverse Lorentz transformations .

Thus, the time interval according to observer in frame S is

$$t = t_2 - t_1 = \frac{t'_2 + \frac{v'x}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t'_1 + \frac{v'x}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t'_2 - t'_1}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{... ... ... (2.58(c))}$$

but $t'_2 - t'_1 = t_0$

Therefore,

$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{... ... ... (2.59)}$$

$\therefore \quad t > t_0$

From above equation, we can say that a clock measure a longer time interval between events occurring in its own frame than the time interval measured by a clock in a frame moving relative to it. In other words, when an observer in motion relative to clock, the time intervals appear to be increased. This phenomenon is called time dilation.

Although time is a relative quantity, we are able to observe the following phenomena :

(i) Time does not run backward for any observer. The sequence of events in a series of events is never altered for any observer.

(ii) No observer can see an event before it takes place.

Let us take some cases to understand time dilation :

1. If $v = 0.1\ c$ then $t = 1.005\ t_0$
2. If $v = 9.5\ c$ then $t = 1.1547\ t_0$
3. If $v = 0.9\ c$ then $t = 2.2942\ t_0$
4. If $v = c$ then $t = \infty$ which is not possible.

Thus, the maximum limit of the velocity of any massive body is c, hence any massive body cannot travel with the velocity of light.

Example 2.4 : A particle with a mean proper life time of 2μs moves through the laboratory with a speed of 0.9 C. Calculate its life line as measured by an observer in the laboratory.

$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad t_0 = 2\ \mu s = 2 \times 10^{-6}\ s, \quad v = 0.9\ c$$

$$\therefore\ t = \frac{2 \times 10^{-6}}{\sqrt{1-\left(\frac{0.9\ c}{c}\right)^2}} = 4.58 \times 10^{-6}\ \text{sec}$$

2.10 VARIATION OF MASS WITH VELOCITY :

According to Newtonian mechanics, the mass of a body does not change with velocity. The same force will produce the same acceleration whether the body is at rest or moving with certain velocity. But according to Eistein, the mass of the body in motion is different from the mass of the body at rest.

Consider two inertial frames of reference S [O - XYZ] and S' [O' - X'Y'Z'], in which S is stationary and S' is moving with velocity *v* in positive X-direction. In order to consider the variation of mass with velocity, we shall consider a collision of two bodies in system (Frame) S' and view it from the frame S. Let two bodies of mass m_1 and m_2 are travelling with velocities u' and -u' parallel to X-axis in the frame S'. Suppose the two bodies collide and after collision coalesce into each other and became one body. Since this collision is elastic so, conservation of momentum hold good.

Using the law of addition of velocities, the velocities u_1 and u_2 in frame S corresponding to u' and -u' are given by

$$u_1 = \frac{u' + v}{1 + \frac{u'v}{c^2}} \text{ and } u_2 = \frac{-u' + v}{1 - \frac{u'v}{c^2}} \qquad \dots\ \dots\ \dots\ (2.60)$$

Further, let the mass of the body travelling with velocity u_1 be m_1 and that of the body moving with velocity u_2 be m_2

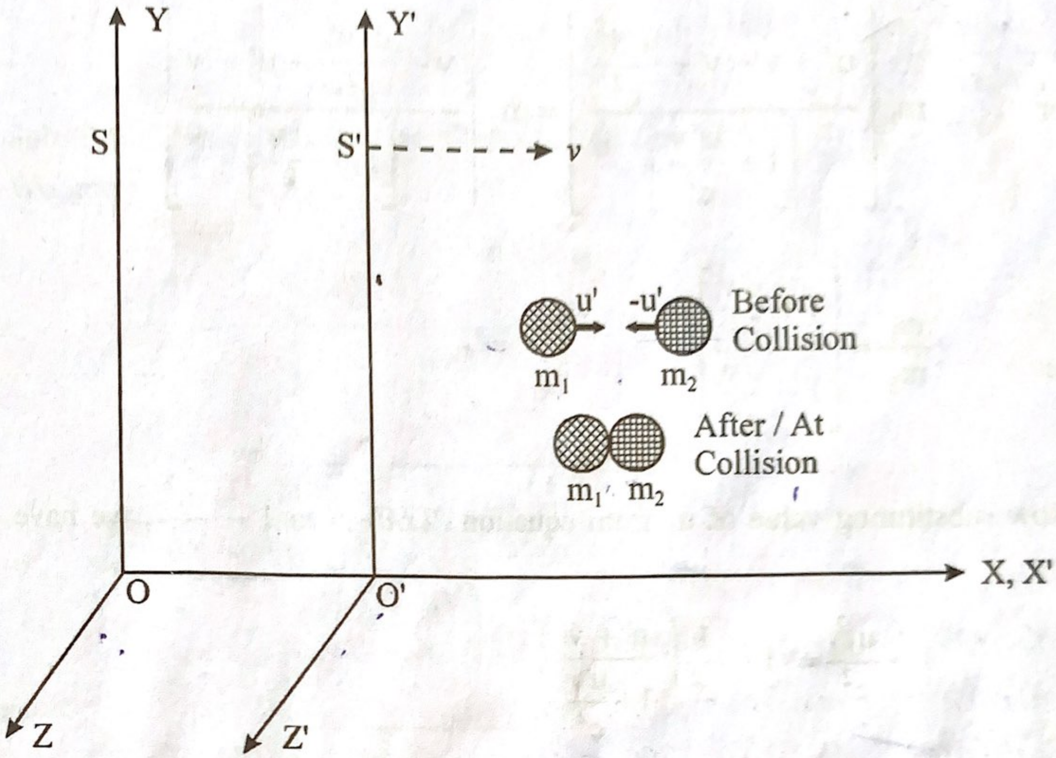


FIG. 2.7

Applying the principle of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v \quad \ldots \ldots \ldots (2.61)$$

because after collision the two bodies are coalesced into one and moving with velocity v as frame S' is moving with velocity v with respect to S.

Substituting the values of u_1 and u_2 in equation 2.61 from equation 2.60, we get

$$m_1\left[\frac{u'+v}{1+\frac{u'v}{c^2}}\right] + m_2\left[\frac{-u'+v}{1-\frac{u'v}{c^2}}\right] = (m_1 + m_2)\,v$$

or
$$m_1\left[\frac{(u'+v)}{\left[1+\frac{u'v}{c^2}\right]} - v\right] = m_2\left[v - \frac{(-u'+v)}{\left[1-\frac{u'v}{c^2}\right]}\right]$$

or $$m_1\left[\frac{u'+v-v-\dfrac{u'v^2}{c^2}}{\left[1+\dfrac{u'v}{c^2}\right]}\right]=m_2\left[\frac{v-\dfrac{u'v^2}{c^2}+u'-v}{\left[1-\dfrac{u'v}{c^2}\right]}\right]$$

or $$\frac{m_1}{m_2}=\left[\frac{1+\dfrac{u'v}{c^2}}{1-\dfrac{u'v}{c^2}}\right] \qquad \ldots \quad \ldots \quad \ldots (2.62)$$

Now substituting value of u_1 from equation (2.60) in to $1-\dfrac{u_1^{\,2}}{c^2}$, we have

$$1-\frac{u_1^{\,2}}{c^2}=1-\frac{1}{c^2}\left(\frac{u'+v}{1+\dfrac{u'v}{c^2}}\right)^2$$

$$1-\frac{u_1^{\,2}}{c^2}=\frac{c^2\left(1+\dfrac{u'v}{c^2}\right)^2-(u'+v)^2}{c^2\left(1+\dfrac{u'v}{c^2}\right)^2} \qquad \ldots \quad \ldots \quad \ldots (2.63)$$

Numerator of equation (2.63) after expansion

$$c^2\left(1+\frac{u'v}{c^2}\right)^2-(u'+v)^2=c^2\left[1+\frac{2u'v}{c^2}+\frac{u'^2v^2}{c^4}\right]-(u'^2+2u'v+v^2)$$

$$=c^2+2u'v+\frac{u'^2v^2}{c^2}-u'^2-2u'v-v^2$$

$$=c^2-u'^2-v^2+\frac{u'^2v^2}{c^2} \quad \text{(by rearranging the terns)}$$

$$=c^2\left(1-\frac{u'^2}{c^2}\right)-v^2\left(1-\frac{u'^2}{c^2}\right)$$

$$=(c^2-v^2)\left(1-\frac{u'^2}{c^2}\right)$$