Study material of B.Sc.(Semester - I)
US01CMTH02
(Radius of Curvature and Rectification)
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US01CMTH02
UNIT-2

1. Curvature

Let $f : I \to \mathbb{R}$ be a sufficiently many times differentiable function on an interval I. Then the points on the graph of $y = f(x)$ is a curve. However, not all curves could be represented as a graph of such a real valued function on intervals viz, the figure of a circle with centre (0,0) and radius 1 in the XY-plane $\mathbb{R}^2$ is one such example of a curve. In this situation, we have to represent the equation of the circle as $x = \cos t$; $y = \sin t$, $t \in [0, 2\pi]$. These are called the *parametric equations* of the circle. Also, let us think of a spring put in $\mathbb{R}^3$. Then the points of this spring is a curve. Thus formally we have the following definition of a curve.

1.1. Definition. Let I be a closed interval and $x = x(t)$, $y = y(t)$ and $z = z(t)$ be real-valued differentiable functions defined on I. Then the points $(x(t), y(t), z(t))$ in the space is called a *locus of the curve* represented by the parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, $t \in I$.

Throughout this chapter we shall be concerned only with curves lying in the XY-plane. For such curves we have $z = 0$. Hence they are described by $x = x(t)$ and $y = y(t)$. A curve lying only in one plane is called a *planer curve*.

1.2. Definition. Let $x = x(t), y = y(t)$ be a curve. If we eliminate t and obtain a relation $g(x, y) = 0$, then this form is called the *cartesian representation of the curve*. Further, if $g(x, y) = 0$ can be written in the form $y = f(x)$ (respectively, $x = f(y)$), then $y = f(x)$ (respectively, $x = f(y)$) is called the *cartesian equation of the curve*.

1.3. Example. Let $I = [0, 1]$ and $x = t$, $y = t^2$. Then this is a curve that can also be represented by the cartesian equation $y = x^2$.

1.4. Definition. Let $y = f(x)$ be a curve. Fix a point A on this curve. For a point P on the curve, let $s = \text{arc}\, AP$ be the arc length from A to P. For a point Q on the curve, let $s + \Delta s = \text{arc}\, AQ$ so that $\Delta s = \text{arc}\, PQ$. Let ℓ_1, ℓ_2 be the tangents to the curve at the points P and Q making angles ψ and $\psi + \Delta\psi$ respectively, with a fixed line in the plane. Clearly, the angle between these two tangents is $\Delta\psi$, called the *total bending* or *total curvature* of the arc between P and Q. Hence the *average bending* or the *average curvature* of the curve between these two points relative to the arc length is given by $\frac{\Delta\psi}{\Delta s}$. The *bending* or the *curvature* of the curve at P is defined to be $\frac{d\psi}{ds} = \lim_{Q\to P} \frac{\Delta\psi}{\Delta s} = \lim_{\Delta s\to 0} \frac{\Delta\psi}{\Delta s}$.

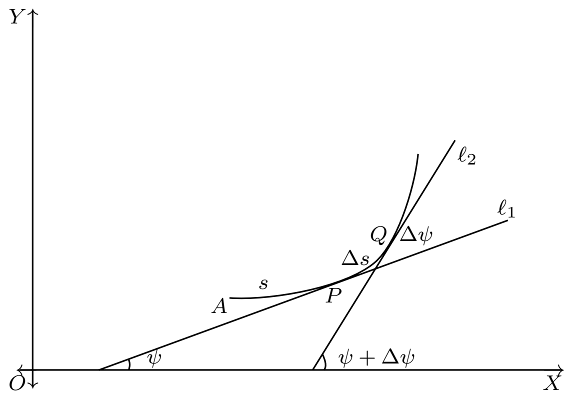

Figure 1.4

2. Derivative of an arc

2.1. Proposition. *Fix a point* $A(x_0, y_0)$ *on a curve given by* $y = f(x)$. *For a point* $P(x, f(x))$ *on the curve, let* s *be the arc length of* arc AP. (*Clearly,* s *is a function of* x.) *Then prove that*

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

PROOF.

Let $y = f(x)$ represent the given curve and A be a fixed point on it. Let $P(x, y)$ be a generic point on the curve. Let the arc $AP = s$. Take a point $Q(x + \Delta x, y + \Delta y)$ on the curve near to P. Let $\operatorname{arc} AQ = s + \Delta s$. From the right angled triangle $\triangle PNQ$, we have,

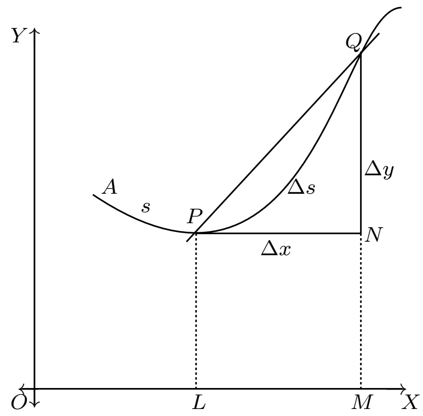

Figure 2.1

$$PQ^2 = PN^2 + NQ^2 = (\Delta x)^2 + (\Delta y)^2$$
$$\Rightarrow \left(\frac{PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$
$$\Rightarrow \left(\frac{\operatorname{chord} PQ}{\operatorname{arc} PQ}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2.$$

Taking $Q \to P$, we get $\operatorname{chord} PQ \to \operatorname{arc} PQ$. Hence,

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \Rightarrow \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

□

The proof of the following corollary is left to the reader.

2.2. Corollary. *Let $x = x(t)$ and $y = y(t)$ be the parametric equations of a curve. Then*

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

2.3. Exercise. In the Proposition 2.1, suppose that the curve is represented by $x = f(y)$. Then deduce that the derivative of the arc length $\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$.

2.4. Definition. Let $h(x, y) = 0$ be a cartesian representation of a curve. By substituting $x = r\cos\theta$ and $y = r\sin\theta$, in this form we get a representation $g(r, \theta) = 0$ of the curve called a *polar representation of the curve.*

We shall be mainly dealing with the form $r = f(\theta)$ of the curve.

2.5. Theorem. *For a polar equation $r = f(\theta)$ of a curve,*

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

PROOF. Let $r = f(\theta)$ represent the given curve and A be a fixed point on it.

Let $P(r, \theta)$ be a generic point on the curve. Let arc $AP = s$. Take a point $Q(r + \Delta r, \theta + \Delta\theta)$ on the curve near to P. Let arc $AQ = s + \Delta s$. From the right angled triangle $\triangle ONP$ as shown in figure, we have,

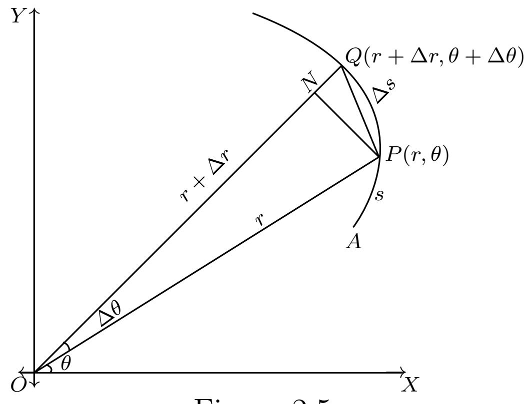

Figure 2.5

$$\sin\Delta\theta = \frac{PN}{OP} = \frac{PN}{r} \Rightarrow PN = r\sin\Delta\theta$$

and

$$\cos\Delta\theta = \frac{ON}{OP} = \frac{ON}{r} \Rightarrow ON = r\cos\Delta\theta.$$

Also, from the figure,

$$\begin{aligned} NQ &= OQ - ON \\ &= r + \Delta r - r\cos\Delta\theta \\ &= r(1 - \cos\Delta\theta) + \Delta r \\ &= 2r\sin^2\frac{\Delta\theta}{2} + \Delta r. \end{aligned}$$

Now from the right angled triangle $\triangle PNQ$, we have,

$$\begin{aligned} &PQ^2 = PN^2 + NQ^2 \\ \Rightarrow &PQ^2 = r^2\sin^2\Delta\theta + (2r\sin^2\frac{\Delta\theta}{2} + \Delta r)^2 \\ \Rightarrow &\left(\frac{PQ}{\Delta\theta}\right)^2 = r^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left[r\sin\left(\frac{\Delta\theta}{2}\right)\left(\frac{\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}\right) + \frac{\Delta r}{\Delta\theta}\right]^2 \\ \Rightarrow &\left(\frac{\text{chord}\,PQ}{\text{arc}\,PQ}\right)^2\left(\frac{\text{arc}\,PQ}{\Delta\theta}\right)^2 = r^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 \\ &\qquad + \left[r\sin\left(\frac{\Delta\theta}{2}\right)\left(\frac{\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}\right) + \frac{\Delta r}{\Delta\theta}\right]^2. \end{aligned}$$

Taking $Q \to P$, we get chord $PQ \to$ arc PQ. Hence,

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 \Rightarrow \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

□

2.6. Example. For the curve $r^n = a^n \cos n\theta$, prove that

$$\frac{ds}{d\theta} = a(\sec n\theta)^{\frac{n-1}{n}}.$$

SOLUTION. Here $r^n = a^n \cos n\theta$. Taking log on both the sides,

$$n \log r = n \log a + \log(\cos n\theta).$$

By differentiating this we get,

$$\begin{aligned} \frac{n}{r}\frac{dr}{d\theta} = -n\frac{\sin n\theta}{\cos n\theta} &\Rightarrow r_1 = \frac{dr}{d\theta} = -r \tan n\theta \\ &\Rightarrow r^2 + r_1^2 = r^2(1 + \tan^2 n\theta) = r^2 \sec^2 n\theta \\ &\Rightarrow \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \\ &\quad = r \sec n\theta = a(\cos n\theta)^{\frac{1}{n}} \sec n\theta = a(\sec n\theta)^{\frac{n-1}{n}}. \end{aligned}$$

□

2.7. Exercise.

1. Show that curvature of a circle is constant and is equal to the reciprocal of its radius.
2. Show that curvature of a straight line is zero.
3. Find $\frac{ds}{dx}$ for the following curves.

   (i) $y = a \cosh \frac{x}{a}$. (ii) $y = a \log\left(\frac{a^2}{a^2-x^2}\right)$.

4. Find $\frac{ds}{dt}$ for the following curves.
   (i) $x = a(t - \sin t)$; $y = a(1 - \cos t)$.
   (ii) $x = a(\cos t + t \sin t)$; $y = a(\sin t - t \cos t)$.
   (iii) $x = ae^t \sin t$; $y = ae^t \cos t$.
5. Find $\frac{ds}{d\theta}$ for the following curves.

   (i) $r = a(1 - \cos\theta)$. (ii) $r^2 = a^2 \cos 2\theta$.

3. Radius of curvature

3.1. Definition. Let P be a point on a curve such that the curvature of the curve at P is nonzero. Then the *radius of the curvature* at P is defined to be the reciprocal of the curvature at P and is denoted by ρ. That is, $\rho = \frac{ds}{d\psi}$.

3.2. Theorem. *Let $y = f(x)$ be a curve and P be a point on it. Then prove that the radius of curvature at P is given by*

$$\rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2},$$

where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$.

PROOF. Let $y = f(x)$ be the given curve. Then $\tan\psi = \frac{dy}{dx}$. Differentiating with respect to s, we get,

$$\begin{aligned}
\sec^2\psi \frac{d\psi}{ds} &= \frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds} = y_2\frac{dx}{ds}\\
\Rightarrow (1+\tan^2\psi)\frac{d\psi}{ds} &= y_2\frac{dx}{ds}\\
\Rightarrow (1+y_1^2)\frac{d\psi}{ds} &= y_2\frac{dx}{ds}\\
\Rightarrow \rho = \frac{ds}{d\psi} &= \frac{1+y_1^2}{y_2}\frac{ds}{dx}\\
\Rightarrow \rho &= \frac{1+y_1^2}{y_2}\sqrt{1+y_1^2}\\
\Rightarrow \rho &= \frac{(1+y_1^2)^{3/2}}{y_2}.
\end{aligned}$$

□

3.3. Theorem. *Let $r = f(\theta)$ be a polar form of a curve with a point P on it. Then prove that the radius of curvature at P is given by*

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2},$$

where $r_1 = f'(\theta)$ and $r_2 = f''(\theta)$.

PROOF. From the figure it is clear that

$\psi = \theta + \varphi$. Hence,

$$\begin{aligned}
\frac{d\psi}{ds} &= \frac{d\theta}{ds} + \frac{d\varphi}{ds}\\
&= \frac{d\theta}{ds} + \frac{d\varphi}{d\theta}\frac{d\theta}{ds}\\
&= \frac{d\theta}{ds}\left(1+\frac{d\varphi}{d\theta}\right). \qquad (3.3.1)
\end{aligned}$$

We know that $\tan\varphi = \frac{r}{r_1}$. Differentiating this with respect to θ, we get,

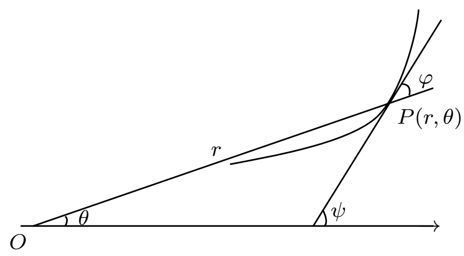

Figure3.3

$$\begin{aligned}
\sec^2\varphi\frac{d\varphi}{d\theta} &= \frac{r_1^2-rr_2}{r_1^2}\\
\Rightarrow \frac{d\varphi}{d\theta} &= \frac{r_1^2-rr_2}{r_1^2}\frac{1}{1+\tan^2\varphi} = \frac{r_1^2-rr_2}{r_1^2}\frac{1}{1+\frac{r^2}{r_1^2}} = \frac{r_1^2-rr_2}{r_1^2+r^2}.
\end{aligned}$$

We also know that $\frac{ds}{d\theta} = \sqrt{r^2+r_1^2}$. Hence by (3.3.1), we get,

$$\frac{d\psi}{ds} = \frac{1}{\sqrt{r^2+r_1^2}}\left(1+\frac{r_1^2-rr_2}{r_1^2+r^2}\right) = \frac{r^2+2r_1^2-rr_2}{(r^2+r_1^2)^{3/2}}.$$

Hence,

$$\rho = \frac{ds}{d\psi} = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}.$$

□

3.4. Example. Prove that if ρ is the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S is its focus, then prove that $\rho^2 \propto SP^3$.

SOLUTION. Let $P(x, y)$ be any point on the give parabola. If the coordinates of the focus S is given by $(a, 0)$, then

$$SP = \sqrt{(x-a)^2 + y^2} = \sqrt{x^2 - 2ax + a^2 + 4ax} = x + a.$$

Now we find ρ for the given parabola $y^2 = 4ax$. Here $2yy_1 = 4a$. That is, $y_1 = \frac{2a}{y}$. Also, $y_2 = -\frac{2a}{y^2}y_1 = -\frac{4a^2}{y^3}$. Hence,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\frac{4a^2}{y^2})^{3/2}}{\frac{-4a^2}{y^3}} = -\frac{(y^2+4a^2)^{3/2}}{4a^2}$$

$$\Rightarrow \rho^2 = \frac{(4ax+4a^2)^3}{16a^4} = \frac{64a^3(x+a)^3}{16a^4} = \frac{4(x+a)^3}{a} = \frac{4}{a}SP^3.$$

This proves that $\rho^2 \propto SP^3$. □

3.5. Example. Show that the radius of curvature at any point of the curve $x = ae^\theta(\cos\theta - \sin\theta)$, $y = ae^\theta(\sin\theta + \cos\theta)$ is twice the perpendicular distance of the tangent at the point form the origin.

SOLUTION. Here

$$\frac{dx}{d\theta} = ae^\theta(\cos\theta - \sin\theta) + ae^\theta(-\sin\theta - \cos\theta) = -2ae^\theta \sin\theta.$$

Similarly,

$$\frac{dy}{d\theta} = 2ae^\theta \cos\theta.$$

Hence $y_1 = \frac{dy}{dx} = -\cot\theta$ and $y_2 = \operatorname{cosec}^2\theta \frac{d\theta}{dx} = \frac{\operatorname{cosec}^3\theta}{-2ae^\theta}$. Thus,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\cot^2\theta)^{3/2}}{-\left(\frac{\operatorname{cosec}^3\theta}{2ae^\theta}\right)} = -2ae^\theta.$$

Now the equation of the tangent at a point is

$$y - ae^\theta(\sin\theta + \cos\theta) = \frac{dy}{dx}(x - ae^\theta(\cos\theta - \sin\theta))$$
$$\Rightarrow y - ae^\theta(\sin\theta + \cos\theta) = -\cot\theta(x - ae^\theta(\cos\theta - \sin\theta))$$
$$\Rightarrow y\sin\theta + x\cos\theta - ae^\theta = 0.$$

Hence the length of the perpendicular distance of the tangent from the origin is $p = \left|\frac{-ae^\theta}{\cos^2\theta + \sin^2\theta}\right| = ae^\theta$. Hence $\rho = -2p$. □

3.6. Example. For the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ prove that $\rho = 4a\cos(\frac{\theta}{2})$. Also show that $\rho_1^2 + \rho_2^2 = 16a^2$, where ρ_1, ρ_2 are the radii of curvature at the points where the tangents are perpendicular.

Solution. $\frac{dx}{d\theta} = a(1 + \cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$. Therefore,

$$y_1 = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\Rightarrow y_2 = \frac{1}{2}\sec^2\frac{\theta}{2}\frac{d\theta}{dx} = \left(\frac{1}{2\cos^2\frac{\theta}{2}}\right)\left(\frac{1}{2a\cos^2\frac{\theta}{2}}\right) = \frac{1}{4a\cos^4\frac{\theta}{2}}.$$

Hence,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \left(1 + \tan^2\frac{\theta}{2}\right)^{3/2} 4a\cos^4\frac{\theta}{2} = 4a\sec^3\frac{\theta}{2}\cos^4\frac{\theta}{2} = 4a\cos(\tfrac{\theta}{2}).$$

If $P(\theta_1)$ and $Q(\theta_2)$ are the points at which the tangents are perpendicular, then $\rho_1 = 4a\cos(\frac{\theta_1}{2})$ and $\rho_2 = 4a\cos(\frac{\theta_2}{2})$. If the tangents at these points make the angles ψ_1 and ψ_2 with the X-axis respectively, then $\tan\psi_1 = \frac{dy}{dx} = \tan\frac{\theta_1}{2}$. Therefore, $\psi_1 = \frac{\theta_1}{2}$. But $\psi_1 - \psi_2 = \frac{\pi}{2}$. Therefore, $\frac{\theta_1}{2} + \frac{\theta_2}{2} = \frac{\pi}{2}$. Hence,

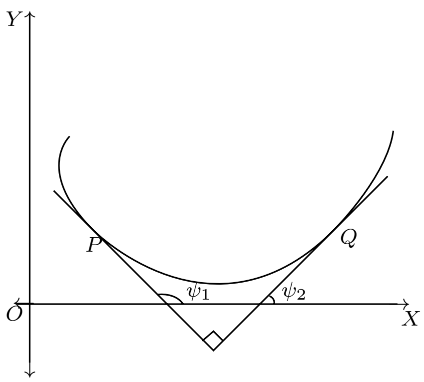

Figure 3.6

$$\rho_1^2 + \rho_2^2 = 16a^2\left[\cos^2\frac{\theta_1}{2} + \cos^2\left(\frac{\pi}{2} - \frac{\theta_1}{2}\right)\right] = 16a^2\left[\cos^2\frac{\theta_1}{2} + \sin^2\frac{\theta_1}{2}\right] = 16a^2.$$

□

3.7. Example. For the curve $r = a(1 - \cos\theta)$, prove that $\rho^2 \propto r$. Also prove that if ρ_1 and ρ_2 are radii of the curvature at the ends of a chord through the pole, $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.

Solution. Here $r_1 = a\sin\theta$ and $r_2 = a\cos\theta$. Hence,

$$\begin{aligned}
\rho &= \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} \\
&= \frac{(a^2(1-\cos\theta)^2 + a^2\sin^2\theta)^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2(1-\cos\theta)\cos\theta} \\
&= \frac{(2a^2(1-\cos\theta))^{3/2}}{3a^2(1-\cos\theta)} \\
&= \frac{(4a^2\sin^2\frac{\theta}{2})^{3/2}}{6a^2\sin^2\frac{\theta}{2}}
\end{aligned}$$

$$= \frac{4}{3} a \sin \tfrac{\theta}{2}.$$

Thus,

$$\rho^2 = \frac{16}{9} a^2 \sin^2 \tfrac{\theta}{2} = \frac{8}{9} a^2 (1 - \cos\theta) = \frac{8ar}{9}$$

$$\Rightarrow \rho^2 \propto r.$$

Let $P(r_1, \theta_1)$ and $P(r_2, \theta_2)$ be the ends of the chord through the pole. Then $\theta_2 - \theta_1 = \pi$. Then $\rho_i^2 = \frac{16}{9} a^2 \sin^2 \frac{\theta_i}{2}$, $(i = 1, 2)$. Hence

$$\begin{aligned} \rho_1^2 + \rho_2^2 &= \frac{16}{9} a^2 \left(\sin^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta_2}{2} \right) \\ &= \frac{16}{9} a^2 \left[\sin^2 \frac{\theta_1}{2} + \sin^2 \left(\frac{\pi + \theta_1}{2} \right) \right] \\ &= \frac{16}{9} a^2 \left(\sin^2 \frac{\theta_1}{2} + \cos^2 \frac{\theta_1}{2} \right) \\ &= \frac{16}{9} a^2. \end{aligned}$$

□

Rectification

4. Derivative of an arc: Revisited

Rectification is the process of computing the length of an arc of a curve. The curves may have different representations – like cartesian, polar and parametric. So, we shall be dealing with all the three forms. Besides, the curve could be expressed as a combination of arcs of two different curves yielding a new closed curve. In this case, the length of arc will be its perimeter. The idea of finding the length of arc is simple. We have obtained the derivative of an arc in Section 2 earlier. It is the derivative of the length of arc s with respect to the independent variable. If we integrate the same, we shall get the length of arc. A curve is said to be *rectifiable* if it is possible to find its length.

5. Length of an arc of a curve

5.1. Theorem. *Let $y = f(x)$ be a cartesian representation of a curve C. Then the length of arc of C between two points A and B corresponding to the x-coordinates a and b respectively, is given by*

$$\text{arc}\, AB = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Proof. Let $s(x)$ be the length of arc of curve between fixed point A on the curve and the generic point $P(x, f(x))$. Then integrating (**??**) from a to b, we have,

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \frac{ds}{dx}\, dx$$

$$\begin{aligned}
&= \int_a^b ds \\
&= \Big[\ s\ \Big]_a^b \\
&= s(b) - s(a) \\
&= \operatorname{arc} AB - \operatorname{arc} AA \\
&= \operatorname{arc} AB.
\end{aligned}$$

□

5.2. Theorem. *Let $r = f(\theta)$ be a polar representation of a curve C. Then the length of arc of C between two points A and B corresponding to the angles $\theta = \theta_0$ and $\theta = \theta_1$ respectively, is given by*

$$\operatorname{arc} AB = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

5.3. Example. Find the length of arc of the parabola $y^2 = 4ax$, $(a > 0)$, measured from the vertex to one extremity of its latus rectum.

Solution. We can write the given equation as $x = \frac{y^2}{4a}$. Then $\frac{dx}{dy} = \frac{y}{2a}$. Therefore,

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{y^2}{4a^2}} = \frac{1}{2a}\sqrt{y^2 + 4a^2}\,.$$

From the figure, we see that coordinates of the vertex O and top end of the latus rectum L are $(0, 0)$ and $(a, 2a)$ respectively. Hence the required length of arc is

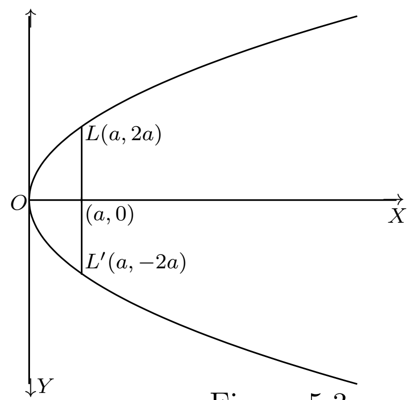

Figure 5.3

$$\begin{aligned}
\operatorname{arc} OL &= \int_0^{2a} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= \frac{1}{2a}\int_0^{2a} \sqrt{y^2 + 4a^2}\, dy \\
&= \frac{1}{2a}\left[\frac{y}{2}\sqrt{y^2 + 4a^2} + \frac{4a^2}{2}\log(y + \sqrt{y^2 + 4a^2})\right]_0^{2a} \\
&= \frac{1}{2a}\left[2\sqrt{2}a^2 + 2a^2\log(2a + 2a\sqrt{2}) - 0 - 2a^2\log 2a\right]
\end{aligned}$$

$$= a\left[\sqrt{2} + \log\left(\frac{2a(1+\sqrt{2})}{2a}\right)\right]$$
$$= a\left(\sqrt{2} + \log(1+\sqrt{2})\right).$$

□

5.4. Example.

(a) Find the entire length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

(b) Prove that the length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ measured from $(0, a)$ to the point (x, y) is given by $\frac{3}{2}(ax^2)^{1/3}$.

Solution. (a)

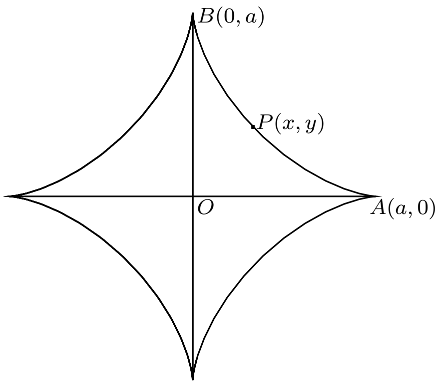

Figure 5.4

Here,

$$x^{2/3} + y^{2/3} = a^{2/3}$$
$$\Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$
$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{y^{2/3}}{x^{2/3}} = \frac{a^{2/3}}{x^{2/3}} = a^{2/3}x^{-2/3}.$$

From the figure, the entire length of the astroid is

$$4 \times \text{arc}\, AB = 4\int_a^0 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= 4\int_a^0 a^{1/3}x^{-1/3}\, dx$$
$$= 4a^{1/3}\left[\frac{x^{2/3}}{2/3}\right]_a^0$$

$$= 4a^{1/3}\left(\frac{-a^{2/3}}{2/3}\right)$$
$$= -6a.$$

Since the length of an arc is always positive, we infer that the entire length of the astroid is $6a$.

(b)

$$\begin{aligned}
\text{The required arc length} &= \text{arc } BP \\
&= \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_0^x a^{1/3}x^{-1/3}\,dx \\
&= a^{1/3}\left[\frac{x^{2/3}}{2/3}\right]_0^x \\
&= a^{1/3}\frac{x^{2/3}}{2/3} \\
&= \frac{3}{2}a^{1/3}x^{2/3} \\
&= \frac{3}{2}(ax^2)^{1/3}.
\end{aligned}$$

□

5.5. Example. Show that the entire length of the curve $x^2(a^2-x^2) = 8a^2y^2$ is $\pi a\sqrt{2}$.

SOLUTION.
The given curve is symmetric about all – X-axis, Y-axis and the origin. Putting $y = 0$, we get $x \in \{0, \pm a\}$. Also since $y^2 = \dfrac{x^2(a^2-x^2)}{8a^2}$, we have $y = \dfrac{x\sqrt{a^2-x^2}}{2\sqrt{2}\,a}$. Hence $-a \le x \le a$ is the only possibility for getting y real. The shape of the given curve is as shown in the figure. It contains two equal loops. Now,

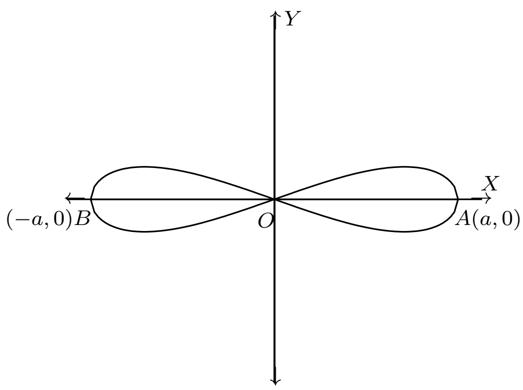

Figure 5.5

$$\begin{aligned}
& 8a^2y^2 = x^2(a^2-x^2) \\
\Rightarrow\quad & 16a^2y\frac{dy}{dx} = 2x(a^2-x^2) + x^2(-2x)
\end{aligned}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{x(a^2-2x^2)}{8a^2y}$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{x(a^2-2x^2)}{8a^2y}\right)^2$$

$$= 1+\frac{x^2(a^2-2x^2)^2}{8a^2x^2(a^2-x^2)}$$

$$= \frac{8a^4-8a^2x^2+a^4-4a^2x^2+4x^4}{8a^2(a^2-x^2)}$$

$$= \frac{(3a^2-2x^2)^2}{8a^2(a^2-x^2)}.$$

From the figure (5.5), we say that

$$\text{the entire length} = 4\,\text{arc}\,OA$$

$$= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\int_0^a \frac{3a^2-2x^2}{2a\sqrt{2}\sqrt{a^2-x^2}}\,dx$$

$$= 4\int_0^a \left[\frac{a^2}{2a\sqrt{2}\sqrt{a^2-x^2}} + 2\frac{a^2-x^2}{2a\sqrt{2}\sqrt{a^2-x^2}}\right]dx$$

$$= 4\int_0^a \left[\frac{a}{2\sqrt{2}\sqrt{a^2-x^2}} + \frac{\sqrt{a^2-x^2}}{a\sqrt{2}}\right]dx$$

$$= 4\left[\frac{a}{2\sqrt{2}}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{a\sqrt{2}}\left(\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)\right]_0^a$$

$$= 4\left[\frac{a}{\sqrt{2}}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2\sqrt{2}a}\right]_0^a$$

$$= 4\frac{a}{\sqrt{2}}\frac{\pi}{2}$$

$$= \pi a\sqrt{2}.$$

□

5.6. Example. Find the length of the cardioid $r = a(1+\cos\theta)$ lying outside the circle $r = -a\cos\theta$.

SOLUTION. First we find the angle between two curves at the point of their intersection.

By comparing them, we get $a(1+\cos\theta) = -a\cos\theta \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \pi \pm \frac{\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. From the figure, we see that the given curve is symmetric about the polar axis and the required arc length is $\operatorname{arc} ABC = 2\operatorname{arc} BA$. Now for the curve $r = a(1+\cos\theta)$,

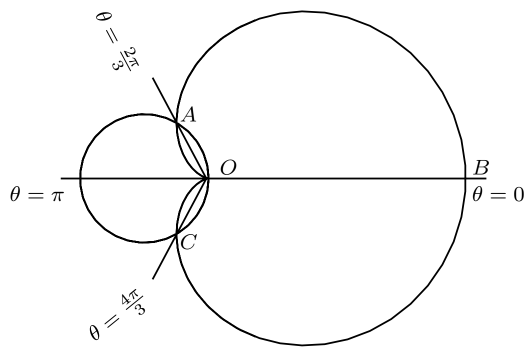

Figure 5.6

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta = 2a^2(1+\cos\theta) = 4a^2\cos^2\frac{\theta}{2}.$$

Hence the required arc length

$$\begin{aligned} 2\operatorname{arc} BA &= 2\int_0^{2\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= 2\int_0^{2\pi/3} 2a\cos(\tfrac{\theta}{2})\, d\theta \\ &= 8a\left[sin\tfrac{\theta}{2}\right]_0^{2\pi/3} \\ &= 8a\left(\sin\tfrac{\pi}{3} - 0\right) = 4a\sqrt{3}. \end{aligned}$$

□

6. Intrinsic equation

6.1. Definition. Let A be a fixed point on a curve C and P be a generic point on the curve. Let $\psi(P)$ denote the angle between the tangents to the curve at points A and P. Also, let $s = \operatorname{arc} AP$. Then the relation between s and ψ is called the *intrinsic equation* of the curve.

It is customary to fix origin (or pole) as the fixed point A if it lies on the curve. Otherwise we mention the fixed point explicitly. We follow this convention throughout this section including exercise. Now we obtain the intrinsic equations of the curve represented in different forms.

I Cartesian form

Let $A(a,b)$ be a fixed point and $P(x,y)$ be a generic point on the curve $y=f(x)$. We develop the equation in a particular case when the tangent to the curve at A is parallel to the X-axis. Then

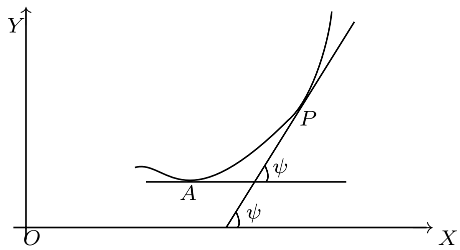

Figure 6.1

$$s=\int_a^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{6.1.1}$$

and

$$\tan\psi=\frac{dy}{dx}. \tag{6.1.2}$$

Eliminating x from (6.1.1) and (6.1.2) we get a relation

$$F(s,\psi)=0,$$

which is the *intrinsic equation* of the curve in cartesian form.

If the curve is represented in the form $x=f(y)$ or in a parametric form, then the intrinsic equation can be obtained similarly by eliminating y or the parameter t respectively. However, in the polar form, the coordinates are changed, so we give intrinsic equation in this case separately.

II Polar form

Let $A(r_1,\theta_1)$ be a fixed point and $P(r,\theta)$ be a generic point on the curve $r=f(\theta)$. We develop the equation in a particular case when the tangent to the curve at A is parallel to the polar axis. Then

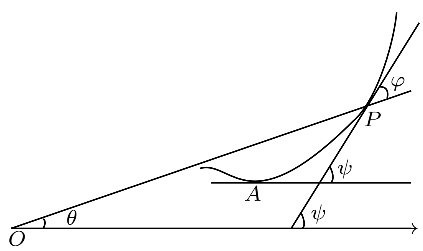

Figure 6.1

$$s=\int_{\theta_1}^{\theta} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta. \tag{6.1.3}$$

Now from the figure,

$$\psi=\theta+\varphi, \tag{6.1.4}$$

where φ is the angle between the radius vector and the tangent at point P. We also know that

$$\tan\varphi=r\frac{d\theta}{dr}. \tag{6.1.5}$$

Eliminating φ and θ from (6.1.3), (6.1.4) and (6.1.5) we get

$$F(s,\psi)=0, \tag{6.1.6}$$

which is the *intrinsic equation* of the curve in polar form.

6.2. Example. Find the intrinsic equation of the Cardioid $r = a(1+\cos\theta)$. Hence prove that $s^2 + 9\rho^2 = 16a^2$, where ρ is the radius of curvature at any point of the curve.

SOLUTION. Here $r = a(1+\cos\theta)$. Therefore, $\frac{dr}{d\theta} = -a\sin\theta$. Hence
$\tan\varphi = r\frac{d\theta}{dr} = -\frac{1+\cos\theta}{\sin\theta} = -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$. Hence,

$$\varphi = \frac{\pi}{2} + \frac{\theta}{2}.$$

Also,

$$\psi = \theta + \varphi = \theta + \frac{\pi}{2} + \frac{\theta}{2} = \frac{3\theta}{2} + \frac{\pi}{2}. \tag{6.2.1}$$

Now,

$$\begin{aligned}
s &= \int_{\theta_1}^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\
&= \int_0^{\theta} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta \\
&= a\int_0^{\theta} \sqrt{2(1+\cos\theta)}\, d\theta \\
&= a\int_0^{\theta} \sqrt{4\cos^2\tfrac{\theta}{2}}\, d\theta \\
&= 2a\int_0^{\theta} \cos\tfrac{\theta}{2}\, d\theta \\
&= 2a\left[2\sin\tfrac{\theta}{2}\right]_0^{\theta} \\
&= 4a\sin\tfrac{\theta}{2} \\
&= 4a\sin\left(\frac{\psi}{3} - \frac{\pi}{6}\right), \qquad \text{(by (6.2.1))} && (6.2.2)
\end{aligned}$$

which is the required intrinsic equation. By differentiating (6.2.2), we have $\rho = \frac{ds}{d\psi} = \frac{4a}{3}\cos\left(\frac{\psi}{3}-\frac{\pi}{6}\right)$. Hence $3\rho = 4a\cos\left(\frac{\psi}{3}-\frac{\pi}{6}\right)$. So, $s^2+9\rho^2 = 16a^2$. □

6.3. Example. Show that the intrinsic equation of the curve $y^3 = ax^2$, is $27s = 8a(\sec^3\psi - 1)$.

SOLUTION. We can write the given equation as $x = \frac{1}{\sqrt{a}}y^{3/2}$. Then $\frac{dx}{dy} = \frac{3}{2}\sqrt{\frac{y}{a}}$. Here the tangent to the curve at the origin is Y-axis. Therefore, $\tan\psi = \frac{dx}{dy}$, That is,

$$\tan\psi = \frac{3}{2}\sqrt{\frac{y}{a}}. \tag{6.3.1}$$

Now,

$$s = \int_0^y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^y \sqrt{1 + \frac{9y}{4a}}\, dy$$

$$= \frac{4a}{9}\,\frac{2}{3}\left[\left(1 + \frac{9y}{4a}\right)^{3/2}\right]_0^y$$

$$\Rightarrow 27s = 8a\left[\left(1 + \frac{9y}{4a}\right)^{3/2} - 1\right]$$

$$\Rightarrow 27s = 8a\left[(1 + \tan^2\psi)^{3/2} - 1\right]$$

$$\Rightarrow 27s = 8a(\sec^3\psi - 1).$$

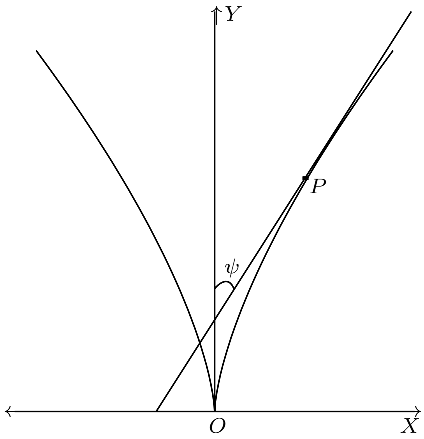

Figure 6.3

□

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