Study material of B.Sc.(Semester - II)

US02CMTH01

(Quadric surfaces)

Prepared by Nilesh Y. Patel

Head,Mathematics Department,V.P.and R.P.T.P.Science College

UNIT-2

Definition 1. Every quadratic equation in x, y and z represents a quadric surface.

To discuss quadric surfaces we have to consider following points :

(1) Symmetry with respect to plane :

- The surface $f(x, y, z) = 0$ is said to be symmetric with respect to the xy- plane, if it remains unchanged on replacing z by $-z$.
- The surface $f(x, y, z) = 0$ is said to be symmetric with respect to the yz- plane, if it remains unchanged on replacing x by $-x$.
- The surface $f(x, y, z) = 0$ is said to be symmetric with respect to the zx- plane, if it remains unchanged on replacing y by $-y$.

(2) Intercepts :

- x-intercept : Put $y = z = 0$
- y-intercept : Put $x = z = 0$
- z-intercept : Put $x = y = 0$

(3) Trace on the plane : The intersection of surface $f(x, y, z) = 0$ with the xy-,yz- and zx- plane is called xy-,yz- and zx- trace respectively on the plane.

- xy-trace : Put $z = 0$
- yz-trace : Put $x = 0$
- zx-trace : Put $y = 0$

(4) Section by planes : The intersection of surface with plane P is called a section of the surface and it is obtained by considering the two equation simultaneously.

Sections by planes which are parallel to the coordinate plane (such as $x = x_1$, $y = y_1, z = z_1$) are vary helpful in visualizing the shape of the surface, we shall consider section by $x = x_1, y = y_1$ and $z = z_1$ planes.

Different Type of quadric surfaces 2.

Ellipsoid : The surface given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is called an ellipsoid.

Elliptic hyperboloid of one sheet : The surface given by any one of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is called an elliptic hyperboloid of one sheet.

Elliptic hyperboloid of two sheet : The surface given by any one of $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is called an elliptic hyperboloid of two sheet.

Elliptic Paraboloid: The surface given by any one of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=cz$, $\frac{x^2}{a^2}+\frac{z^2}{c^2}=by$ and $\frac{y^2}{b^2}+\frac{z^2}{c^2}=ax$ is called an elliptic Paraboloid.

hyperbolic Paraboloid : The surface given by any one of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=cz$, $\frac{x^2}{a^2}-\frac{z^2}{c^2}=by$ and $\frac{y^2}{b^2}-\frac{z^2}{c^2}=ax$ is called a hyperbolic Paraboloid.

Elliptic cone : The surface given by any one of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$, $\frac{x^2}{a^2}+\frac{z^2}{c^2}=\frac{y^2}{b^2}$ and $\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x^2}{a^2}$ is called an elliptic cone.

Examples 3. Identify, describe and sketch the surface.

(1) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1; (b \geq a, c). -\,-\,-\,-\,-\,- (*)$

Solution : Here given surface is ellipsoid.

(i) Symmetry w.r.t plane : Here all powers of x, y & z are even power. Therefore it is symmetric w.r.t xy-plane , yz-plane and zx-plane.

(ii) Intercepts :

x-int : Put $y=z=0$ we get $\frac{x^2}{a^2}=1 \Rightarrow x^2=a^2 \Rightarrow \boxed{x=\pm a}$

y-int : Put $x=z=0$ we get $\frac{y^2}{b^2}=1 \Rightarrow y^2=b^2 \Rightarrow \boxed{y=\pm b}$

z-int : Put $x=y=0$ we get $\frac{z^2}{c^2}=1 \Rightarrow z^2=c^2 \Rightarrow \boxed{z=\pm c}$

(iii) Trace on the Planes :

xy-trace : Put $z=0$, we get $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Which is ellipse.

yz-trace : Put $x=0$, we get $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Which is ellipse.

zx-trace : Put $y=0$, we get $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$. Which is ellipse.

(iv) Section by planes :

Section by $x=x_1$ planes : Put $x=x_1$ in $(*)$, we get

$$\frac{x_1^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \Rightarrow \frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{x_1^2}{a^2} \Rightarrow \frac{y^2}{b^2}+\frac{z^2}{c^2}=k,$$

where $k = 1 - \dfrac{x_1^2}{a^2}$.

$\Rightarrow \dfrac{y^2}{kb^2} + \dfrac{z^2}{kc^2} = 1$, which is ellipse, if $k > 0$ i.e. if $1 - \dfrac{x_1^2}{a^2} > 0$ i.e. if $-a < x_1 < a$. Thus section by $x = x_1$ planes is an ellipse.

Section by $y = y_1$ planes : Put $y = y_1$ in $(*)$, we get

$\dfrac{x^2}{a^2} + \dfrac{y_1^2}{b^2} + \dfrac{z^2}{c^2} = 1 \Rightarrow \dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = 1 - \dfrac{y_1^2}{b^2} \Rightarrow \dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = k$,

where $k = 1 - \dfrac{y_1^2}{b^2}$.

$\Rightarrow \dfrac{x^2}{ka^2} + \dfrac{z^2}{kc^2} = 1$, which is ellipse, if $k > 0$ i.e. if $1 - \dfrac{y_1^2}{b^2} > 0$ i.e. if $-b < y_1 < b$. Thus section by $y = y_1$ planes is an ellipse.

Section by $z = z_1$ planes : Put $z = z_1$ in $(*)$, we get

$\dfrac{x_1^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z_1^2}{c^2} = 1 \Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 - \dfrac{z_1^2}{c^2} \Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = k$,

where $k = 1 - \dfrac{z_1^2}{c^2}$.

$\Rightarrow \dfrac{x^2}{ka^2} + \dfrac{y^2}{kb^2} = 1$, which is ellipse, if $k > 0$ i.e. if $1 - \dfrac{z_1^2}{c^2} > 0$ i.e. if $-c < z_1 < c$. Thus section by $z = z_1$ planes is an ellipse.

(2) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1. - - - - - - (*)$

Solution : Here given surface is elliptic hyperboloid of one sheet.

(i) Symmetry w.r.t plane : Here all powers of x, y & z are even power. Therefore it is symmetric w.r.t xy-plane , yz-plane and zx-plane.

(ii) Intercepts :

x-int : Put $y = z = 0$ we get $\dfrac{x^2}{a^2} = 1 \Rightarrow x^2 = a^2 \Rightarrow \boxed{x = \pm a}$

y-int : Put $x = z = 0$ we get $\dfrac{y^2}{b^2} = 1 \Rightarrow y^2 = b^2 \Rightarrow \boxed{y = \pm b}$

z-int : Put $x = y = 0$ we get $\dfrac{z^2}{c^2} = 1 \Rightarrow z^2 = -c^2 \Rightarrow \boxed{\text{not possible}}$

(iii) Trace on the Planes :

xy**-trace :** Put $z = 0$, we get $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Which is ellipse.

yz**-trace :** Put $x = 0$, we get $\dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$. Which is hyperbola.

zx**-trace :** Put $y = 0$, we get $\dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1$. Which is hyperbola.

(iv) Section by planes :

Section by $x = x_1$ **planes :** Put $x = x_1$ in $(*)$, we get
$\dfrac{x_1^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1 \Rightarrow \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{x_1^2}{a^2} \Rightarrow \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = k$, where $k = 1 - \dfrac{x_1^2}{a^2}$.
$\Rightarrow \dfrac{y^2}{kb^2} - \dfrac{z^2}{kc^2} = 1$, which is hyperbola, if $k > 0$ i.e. if $1 - \dfrac{x_1^2}{a^2} > 0$ i.e. if $-a < x_1 < a$. Thus section by $x = x_1$ planes is an hyperbola.

Section by $y = y_1$ **planes :** Put $y = y_1$ in $(*)$, we get
$\dfrac{x^2}{a^2} + \dfrac{y_1^2}{b^2} - \dfrac{z^2}{c^2} = 1 \Rightarrow \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{y_1^2}{b^2} \Rightarrow \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = k$, where $k = 1 - \dfrac{y_1^2}{b^2}$.
$\Rightarrow \dfrac{x^2}{ka^2} - \dfrac{z^2}{kc^2} = 1$, which is hyperbola, if $k > 0$ i.e. if $1 - \dfrac{y_1^2}{b^2} > 0$ i.e. if $-b < y_1 < b$. Thus section by $y = y_1$ planes is an hyperbola.

Section by $z = z_1$ **planes :** Put $z = z_1$ in $(*)$, we get
$\dfrac{x_1^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z_1^2}{c^2} = 1 \Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \dfrac{z_1^2}{c^2} \Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = k$, where $k = 1 + \dfrac{z_1^2}{c^2}$.
$\Rightarrow \dfrac{x^2}{ka^2} + \dfrac{y^2}{kb^2} = 1$, which is ellipse, if $k = 1 + \dfrac{z_1^2}{c^2} > 0$. Thus section by $z = z_1$ planes is an ellipse.

(3) $-\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1.------(*)$

Solution : Here given surface is elliptic hyperboloid of two sheet.

(i) Symmetry w.r.t plane : Here all powers of x, y & z are even power. Therefore it is symmetric w.r.t xy-plane , yz-plane and zx-plane.

(ii) Intercepts :

x-int : Put $y=z=0$ we get $\dfrac{x^2}{a^2}=1 \Rightarrow x^2=-a^2 \Rightarrow \boxed{\text{not possible}}$

y-int : Put $x=z=0$ we get $\dfrac{y^2}{b^2}=1 \Rightarrow y^2=b^2 \Rightarrow \boxed{y=\pm b}$

z-int : Put $x=y=0$ we get $\dfrac{z^2}{c^2}=1 \Rightarrow z^2=-c^2 \Rightarrow \boxed{\text{not possible}}$

(iii) Trace on the Planes :

xy**-trace :** Put $z=0$, we get $-\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Which is hyperbola.

yz**-trace :** Put $x=0$, we get $\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$. Which is hyperbola.

zx**-trace :** Put $y=0$, we get $\dfrac{x^2}{a^2}+\dfrac{z^2}{c^2}=-1$. Which is not possible.

(iv) Section by planes :

Section by $x=x_1$ **planes :** Put $x=x_1$ in $(*)$, we get

$-\dfrac{x_1^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1 \Rightarrow \dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1+\dfrac{x_1^2}{a^2} \Rightarrow \dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=k$, where $k=1-\dfrac{x_1^2}{a^2}$.

$\Rightarrow \dfrac{y^2}{kb^2}-\dfrac{z^2}{kc^2}=1$. Thus section by $x=x_1$ planes is an hyperbola.

Section by $y=y_1$ **planes :** Put $y=y_1$ in $(*)$, we get

$-\dfrac{x^2}{a^2}+\dfrac{y_1^2}{b^2}-\dfrac{z^2}{c^2}=1 \Rightarrow \dfrac{x^2}{a^2}+\dfrac{z^2}{c^2}=\dfrac{y_1^2}{b^2}-1 \Rightarrow \dfrac{x^2}{a^2}+\dfrac{z^2}{c^2}=k$, where $k=\dfrac{y_1^2}{b^2}-1$.

$\Rightarrow \dfrac{x^2}{ka^2}+\dfrac{z^2}{kc^2}=1$, which is hyperbola, if $k>0$ i.e. if $\dfrac{y_1^2}{b^2}1-1>0$ i.e. if $-b>y_1>b$. Thus section by $y=y_1$ planes is an ellipse.

Section by $z=z_1$ **planes :** Similarly section by $z=z_1$ planes is

$-\dfrac{x^2}{ka^2}+\dfrac{y^2}{kb^2}=1$. Thus section by $z=z_1$ planes is an hyperbola.

(4) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = cz; (c > 0). - - - - - - (*)$

Solution : Here given surface is hyperbolic Paraboloid.

(i) Symmetry w.r.t plane : Here all powers of x and y are even power. Therefore it is symmetric w.r.t yz-plane and zx-plane. It is not symmetric w.r.t xy-plane.

(ii) Intercepts :

x-int : Put $y = z = 0$ we get $\dfrac{x^2}{a^2} = 0 \Rightarrow x^2 = 0 \Rightarrow \boxed{x = 0}$

y-int : Put $x = z = 0$ we get $\dfrac{-y^2}{b^2} = 0 \Rightarrow -y^2 = 0 \Rightarrow \boxed{y = 0}$

z-int : Put $x = y = 0$ we get $cz = 1 \Rightarrow cz = 0 \Rightarrow \boxed{z = 0}$

(iii) Trace on the Planes :

xy**-trace :** Put $z = 0$, we get $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} = 0 \Rightarrow x = \pm\dfrac{a}{b}y$.

Thus xy-trace is pair of lines through the origin.

yz**-trace :** Put $x = 0$, we get $\dfrac{-y^2}{b^2} = cz \Rightarrow y^2 = -b^2cz$. Which is parabola.

zx**-trace :** Put $y = 0$, we get $\dfrac{x^2}{a^2} = cz \Rightarrow x^2 = a^2cz$. Which is parabola.

(iv) Section by planes :

Section by $x = x_1$ planes : Put $x = x_1$ in $(*)$, we get

$\dfrac{x_1^2}{a^2} - \dfrac{y^2}{b^2} = cz \Rightarrow y^2 = -b^2\left[cz - \dfrac{x_1^2}{a^2}\right]$. Which is parabola.

Section by $y = y_1$ planes : Put $y = y_1$ in $(*)$, we get

$\dfrac{x^2}{a^2} - \dfrac{y_1^2}{b^2} = cz \Rightarrow x^2 = a^2\left[cz + \dfrac{y_1^2}{b^2}\right]$. Which is parabola.

Section by $z = z_1$ planes : Put $z = z_1$ in $(*)$, we get

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = cz_1 \Rightarrow \dfrac{x^2}{ka^2} - \dfrac{y^2}{kb^2} = 1$. Which is hyperbola, When $z_1 > 0$.

(5) $\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = \dfrac{y^2}{b^2}. - - - - - - (*)$

Solution : Here given surface is elliptic cone.

(i) Symmetry w.r.t plane : Here all powers of x, y & z are even power. Therefore it is symmetric w.r.t xy-plane , yz-plane and zx-plane.

(ii) Intercepts :

x-int : Put $y = z = 0$ we get $\dfrac{x^2}{a^2} = 0 \Rightarrow x^2 = 0 \Rightarrow \boxed{x = 0}$

y-int : Put $x = z = 0$ we get $\dfrac{y^2}{b^2} = 0 \Rightarrow y^2 = 0 \Rightarrow \boxed{y = 0}$

z-int : Put $x = y = 0$ we get $\dfrac{z^2}{c^2} = 0 \Rightarrow z^2 = 0 \Rightarrow \boxed{z = 0}$

(iii) Trace on the Planes :

xy-**trace :** Put $z = 0$, we get $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} \Rightarrow \dfrac{x}{a} = \pm\dfrac{y}{b}$. Which is pair of lines through the origin.

yz-**trace :** Put $x = 0$, we get $\dfrac{y^2}{b^2} = \dfrac{z^2}{c^2} \Rightarrow \dfrac{z}{c} = \pm\dfrac{y}{b}$. Which is pair of lines through the origin.

zx-**trace :** Put $y = 0$, we get $\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = 0 \Rightarrow x = z = 0$. Thus zx-trace is (0,0,0).

(iv) Section by planes :

Section by $x = x_1$ **planes :** Put $x = x_1$ in $(*)$, we get
$\dfrac{x_1^2}{a^2} + \dfrac{z^2}{c^2} = \dfrac{y^2}{b^2} \Rightarrow \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = \dfrac{x_1^2}{a^2} \Rightarrow \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = k$, where $k = \dfrac{x_1^2}{a^2}$.
$\Rightarrow \dfrac{y^2}{kb^2} - \dfrac{z^2}{kc^2} = 1$. Which is hyperbola.

Section by $y = y_1$ **planes :** Put $y = y_1$ in $(*)$, we get
$\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = \dfrac{y_1^2}{b^2} \Rightarrow \dfrac{x^2}{ka^2} + \dfrac{z^2}{kc^2} = 1$ where $k = 1 - \dfrac{y_1^2}{b^2}$. Which is ellipse if $k > 0$.

Section by $z = z_1$ **planes :** Put $z = z_1$ in $(*)$, we get
$\dfrac{x^2}{a^2} + \dfrac{z_1^2}{c^2} = \dfrac{y^2}{b^2} \Rightarrow \dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = \dfrac{z_1^2}{c^2} \Rightarrow \dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = k$, where $k = \dfrac{z_1^2}{c^2}$.
$\Rightarrow \dfrac{y^2}{kb^2} - \dfrac{x^2}{ka^2} = 1$. Which is hyperbola.

Examples 4. Identify, describe and sketch the surface.

(1) $\dfrac{x^2}{9} - \dfrac{y^2}{16} - \dfrac{z^2}{9} = 1 - - - - (*)$

Solution : Here given surface is elliptic hyperboloid of two sheet.

(i) Symmetry w.r.t plane : Here all powers of x, y and z are even power. Therefore it is symmetric w.r.t xy-plane, yz-plane and zx-plane.

(ii) Intercepts :

x-int : Put $y = z = 0$ we get $\dfrac{x^2}{9} = 1 \Rightarrow x^2 = 9 \Rightarrow \boxed{x = \pm 3}$

y-int : Put $x = z = 0$ we get $\dfrac{-y^2}{16} = 1 \Rightarrow y^2 = -16 \Rightarrow \boxed{\text{not possible}}$

z-int : Put $x = y = 0$ we get $\dfrac{-z^2}{9} = 1 \Rightarrow z^2 = -9 \Rightarrow \boxed{\text{not possible}}$

(iii) Trace on the Planes :

xy**-trace :** Put $z = 0$, we get $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$. Which is hyperbola.

yz**-trace :** Put $x = 0$, we get $-\dfrac{y^2}{16} - \dfrac{z^2}{9} = 1 \Rightarrow \dfrac{y^2}{16} + \dfrac{z^2}{9} = -1$. Which is not possible.

zx**-trace :** Put $y = 0$, we get $\dfrac{x^2}{9} - \dfrac{z^2}{9} = 1$. Which is hyperbola.

(iv) Section by planes :

Section by $x = x_1$ planes : Put $x = x_1$ in $(*)$, we get
$\dfrac{x_1^2}{9} - \dfrac{y^2}{16} - \dfrac{z^2}{9} = 1 \Rightarrow \dfrac{y^2}{16} + \dfrac{z^2}{9} = \dfrac{x_1^2}{9} - 1 \Rightarrow \dfrac{y^2}{k16} + \dfrac{z^2}{k9} = 1$ which is ellipse, when $k = \dfrac{x_1^2}{9} - 1 > 0$, i.e when $x_1^2 > 9$,i.e when $-3 > x_1 > 3$.

Section by $y = y_1$ planes : Put $y = y_1$ in $(*)$, we get
$\dfrac{x^2}{9} - \dfrac{y_1^2}{16} - \dfrac{z^2}{9} = 1 \Rightarrow \dfrac{x^2}{9} - \dfrac{z^2}{9} = 1 + \dfrac{y_1^2}{16} \Rightarrow \dfrac{x^2}{k9} - \dfrac{z^2}{k9} = 1$. Which is hyperbola.

Section by $z = z_1$ planes : Put $z = z_1$ in $(*)$, we get
$\dfrac{x^2}{9} - \dfrac{y^2}{16} - \dfrac{z_1^2}{9} = 1 \Rightarrow \dfrac{x^2}{9} - \dfrac{y^2}{16} = 1 + \dfrac{z_1^2}{9} \Rightarrow \dfrac{x^2}{k9} - \dfrac{y^2}{k16} = 1$. Which is hyperbola.

(2) $\dfrac{y^2}{4} - \dfrac{z^2}{1} = 2x$

Solution : Here given surface is hyperbolic paraboloid.

(i) Symmetry w.r.t plane : Here all powers of y and z are even power. Therefore it is symmetric w.r.t xy-plane and zx-plane. It is not symmetric w.r.t yz-plane.

(ii) Intercepts :

x-int : Put $y = z = 0$ we get $2x = 0 \Rightarrow \boxed{x = 0}$

y-int : Put $x = z = 0$ we get $\dfrac{y^2}{4} = 0 \Rightarrow y^2 = 0 \Rightarrow \boxed{\text{y} = 0}$

z-int : Put $x = y = 0$ we get $\dfrac{z^2}{1} = 0 \Rightarrow z^2 = 0 \Rightarrow \boxed{\text{z} = 0}$

(iii) Trace on the Planes :

xy-**trace :** Put $z = 0$, we get $\dfrac{y^2}{4} = 2x \Rightarrow y^2 = 8x$. Thus xy-trace is parabola .

yz-**trace :** Put $x = 0$, we get $\dfrac{y^2}{4} - \dfrac{z^2}{1} = 0 \Rightarrow y = \pm 2z$. Which is pair of lines passing through the origin .

zx-**trace :** Put $y = 0$, we get $-\dfrac{z^2}{1} = 2x \Rightarrow z^2 = -2x$ Which is Parabola.

(iv) Section by planes :

Section by $x = x_1$ **planes :** Put $x = x_1$ in $(*)$, we get $\dfrac{y^2}{4} - \dfrac{z^2}{1} = 2x_1 \Rightarrow \dfrac{y^2}{4} - \dfrac{z^2}{1} = k \Rightarrow \dfrac{y^2}{4k} - \dfrac{z^2}{k} = 1$, where $k = 2x_1$. Which is hyperbola.

Section by $y = y_1$ **planes :** Put $y = y_1$ in $(*)$, we get $\dfrac{y_1^2}{4} - \dfrac{z^2}{1} = 2x \Rightarrow -\dfrac{z^2}{1} = 2x - \dfrac{y_1^2}{4} \Rightarrow z^2 = -\left[2x - \dfrac{y_1^2}{4}\right]$. Which is Parabola.

Section by $z = z_1$ **planes :** Put $z = z_1$ in $(*)$, we get $\dfrac{y^2}{4} - \dfrac{z_1^2}{1} = 2x \Rightarrow \dfrac{y^2}{4} = 2x + \dfrac{z_1^2}{1} \Rightarrow y^2 = 4\left[2x + \dfrac{z_1^2}{1}\right]$. which is Parabola.

Example 5. Identify the given surface $9x^2 + 4y^2 - 9z^2 - 18x - 8y - 18z = 32$.

Solution : Here, $9x^2 + 4y^2 - 9z^2 - 18x - 8y - 18z = 32$.

$$\Rightarrow \quad 9(x^2 - 2x) + 4(y^2 - 2y) - 9(z^2 + 2z) = 32$$

$$\Rightarrow \quad 9(x^2 - 2x + 1 - 1) + 4(y^2 - 2y + 1 - 1) - 9(z^2 + 2z + 1 - 1) = 32$$

$$\Rightarrow \quad 9(x - 1)^2 - 9 + 4(y - 1)^2 - 4 - 9(z + 1)^2 + 9 = 32$$

$$\Rightarrow \quad 9(x - 1)^2 + 4(y - 1)^2 - 9(z + 1)^2 = 36$$

$$\Rightarrow \quad \frac{(x-1)^2}{4} + \frac{(y-1)^2}{9} - \frac{(z+1)^2}{4} = 1$$

Translating the origin to $(1, 1, -1)$ the equation of the surface in the new system becomes

$$\frac{x'^2}{4} + \frac{y'^2}{9} - \frac{z'^2}{4} = 1$$

Which is elliptic hyperboloid of one sheet.

Example 6. Show that $Ax^2+By^2+Cz^2 = D$ represents an elliptic hyperboloid of one sheet if one coefficient is negative and $D > 0$.

Solution : Suppose $A < 0$, $B > 0$, $C > 0$ and given $D > 0$. Let $A' = -A > 0$ then given equation becomes $-A'x^2 + By^2 + Cz^2 = D$

$$-\frac{x^2}{\frac{D}{A'}} + \frac{y^2}{\frac{D}{B}} + \frac{x^2}{\frac{D}{C}} = 1 - - - - - - -(*)$$

where, $\frac{D}{A'}, \frac{D}{B}, \frac{D}{C}$ are all positive.

Let $\frac{D}{A'} = a^2, \frac{D}{B} = b^2, \frac{D}{C} = c^2$, then by $(*)$ $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{x^2}{c^2} = 1$. Which is elliptic hyperboloid of one sheet.

Describe sperical polar coordinate 7. The frame of reference consists of a point O (the pole or origin) and two mutually perpendicular rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ originating from it. Let α be the plane containing $\overrightarrow{OA}$ and perpendicular to $\overrightarrow{OB}$. Then a point P is given by the coordinates (ρ, θ, ϕ),where

$$\rho = OP, \qquad \rho \geq 0$$

$$\theta = \angle(\overrightarrow{OA}, \text{ Projection of } \overrightarrow{OP} \text{ on } \alpha), \quad 0 \leq \theta \leq 2\pi$$

and

$$\phi = \angle(\overrightarrow{OB}, \overrightarrow{OP}), \quad 0 \leq \phi \leq \pi$$

To determine the coordinates of P, take $PM\perp$ plane α. Then figure (a)

$$\rho = OP,\ \theta = \angle AOM,\ \phi = \angle BOP = \angle OPM.$$

To plot a point $P(\rho, \theta, \phi)$, make an angle of measure θ at O, with $\overrightarrow{OA}$ as the initial side; on the terminal side $\overrightarrow{OM}$, make an angle of measure $\frac{\pi}{2} - \phi$ at O, in the plane perpendicular to α, that is, plane MOB; on the terminal side of this angle, take ρ units to get $P(\rho, \theta, \phi)$.

The relevance of the system with a sphere is now very clear. Let C denotes the circle of intersection of the sphere, with center O and radius ρ, and the plane α. Suppose C intersects $\overrightarrow{OA}$ in R. Then, to locate $P(\rho, \theta, \phi)$. We move from R then move along C to go to S, such that $\angle ROS = \theta$; move along the great circle SB. Now, to get P, by taking $\angle SOP = \frac{\pi}{2} - \phi$. Thus variation in θ and ϕ give us. Point on the sphere with center O and radius ρ. Hence the name spherical polar coordinates.

Example 8. Plot the point $(3, 30^0, 90^0)$ and $(2, 7\pi/4, \pi/6)$.

Example 9. By proper choice of axes, the cartesian coordinate (x, y, z) of a point can be expressed in terms of spherical coordinates (ρ, θ, ϕ) as

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.$$

Solution : Let three mutually perpendicular lines XOX', YOY' and ZOZ' be the x,- y- and z- axes respectively. In the polar frame of reference, choose the pole at origin and the two axes along the positive x- and z- axes. Then the plane α is the xy - plane (figure (a)). Let P be the point whose Cartesian and polar coordinate are (x, y, z) and (ρ, θ, ϕ) respectively.Let $PM\perp$ plane α and $MN\perp$ x- axes. Then,

$$OP = \rho, \quad \angle OPM = \phi, \quad \text{and} \quad \angle MON = \theta.$$

Form right $\triangle OMP$, $OM = \rho \sin \phi$, $MP = \rho \cos \phi$ and
form right $\triangle ONM$, $MN = \rho \sin \phi \sin \theta$, $ON = \rho \sin \phi \cos \theta$.

$$\therefore\ x = ON = \rho \sin \phi \cos \theta, \quad y = MN = \rho \sin \phi \sin \theta, \quad z = MP = \rho \cos \phi.$$

Example 10. Prove that the equation $r = a$, a positive constant represent a sphere with center at O and radius a.

Solution : Let $P(\rho, \theta, \phi)$ be a point for which $r = a$. Then $OP = a$.

$$\Rightarrow P \text{ is a constant distance } a \text{ from the pole } O.$$

$$\Rightarrow\ P \text{ lies on a sphere with centre at } O \text{ and radius } a.$$

Example 11. Prove that $\theta = \beta, \beta$ a constant, $\beta \in [0, 2\pi)$, is a half - plane perpendicular to α and containing $\overrightarrow{OC}$, where $C \in \alpha$ such that $\angle AOC = \beta$.

Solution : Let $P(\rho, \theta, \phi)$ be any point on the surface $\theta = \beta$.
If $PM \perp \alpha$, then $\angle AOM = \beta$ for every position of P.
But there is no restriction on ρ or ϕ. If $\angle AOC = \beta$, then our set consists of all points whose projection on α lies on $\overrightarrow{OC}$ shown in figure( ). Hence it is set containing all points on the half-plane perpendicular to α and containing $\overrightarrow{OC}$. In given figure ( ), it is the half-plane OCB.

Example 12. Prove that $\phi = \gamma, \gamma$ a constant, $\gamma \in (\theta, \pi/2)$, is the upper half of cone, with vertex at O, axis along the positive z-axis and the semivertical angle γ.

Solution : Take any point $P(r, \theta, \phi)$, with $\phi = \gamma$. $\therefore$ P is (r, θ, γ).

Now γ is fixed. Keep θ also fixed and let r vary. It gives the ray $\overrightarrow{OP}$. Now let θ also vary, so that only $\phi = \gamma$ is fixed. The ray $\overrightarrow{OP}$ will rotate about the z-axis (fig.( )) always making an angle with it.

Hence the surface consists of at rays $\overrightarrow{OP}$ making an angle γ with $\overrightarrow{OZ}$. That is, we have the upper half of the right circular cone, whose vertex is at O, axis along $\overrightarrow{OZ}$ and semi vertical angle γ.

Describe Cylindric polar coordinate 13. The frame of reference consists of to mutually perpendicular ray $\overrightarrow{OA}$ and $\overrightarrow{OB}$, originating form a common point O. Let α be the plane containing $\overrightarrow{OA}$ and perpendicular to $\overrightarrow{OB}$. Then a point P in space, is assigned the coordinates (ρ, θ, z), where, given figure and

$$\rho = \text{ projection of } \overrightarrow{OP} \text{ on } \alpha, \qquad \rho > 0.$$

$$\theta = (\overrightarrow{OA}, \text{projection of } \overrightarrow{OP} \text{ on } \alpha), \quad 0 \leq \theta \leq 2\pi$$

and

$$z = \text{oriented } d(P, \alpha), \quad -\infty \leq z \leq \infty$$

To determine the coordinate of P, take $PM\perp$ plane α. Then, form figure ( )

$$P = OM, \quad \theta = \angle AOM, \quad z = MP.$$

Example 14. Plot the points $(2, \pi/4, 3)$ and $(2, \pi/4, -3)$.

Solution :

Draw $\angle AOR = \dfrac{\pi}{4}$ in the anticlockwise direction form $\overrightarrow{OA}$ shown in given figure : ( ). On $\overrightarrow{OR}$ take $M \ni OM = 2$. Take line $l \| OB$, choose P and Q on $l \ni MP = 3$ units upwardsand $MQ = 3$ units downwards.

Figure ( ) shows $P(2, \pi/4, 3)$ and $Q(2, \pi/4, -3)$.

Example 15. Describe the surfaces given by (i) $\rho = 3$ (ii) $z = 4$ (iii) $\theta = 0$.

Solution : (i) $\rho = 3$.

Any point P on the surface is $(3, \theta, z)$. This means that whatever the θ, the projection M of P on α has to be such that $OM = 3$.

That is, the projection M of P on α lies on the circle about O, with radius 3.

As z varies now, we get points on the right circular cylinder, whose base is this circle.

(ii) $z = 4$

Any point of the surface is $(\rho, \theta, 4)$. As ρ and θ vary, the point of projection can be any point of the plane α. Since $z = 4$, fixed, the point of the surface is always at a distance for form α, and above it. So the surface is a plane parallel to α, at a distance of 4 units above it figure ( ).

(iii) $\theta = 0$

Any point of the surface is $P(\rho, 0, z)$. Now $\theta = 0$. Hence the points of projection M of P on α, must lie on $\overrightarrow{OA}$ figure: ( ). So the points F of the surface must lie in the vertical half-plane OAB.

This gives the half-plane OAB containing $\overrightarrow{OA}$, or, half of the zx-plane with $x \geq 0$.

Example 16. Find Jacobian of $x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$

Solution :