Study material of B.Sc.(Semester - I)
US01CMTH02
(Partial Derivatives)
Prepared by Nilesh Y. Patel
Head,Mathematics Department,V.P.and R.P.T.P.Science College

US01CMTH02
UNIT-3

1. Partial derivatives

The concept of partial derivative plays a vital role in differential calculus. The different ways of limit discussed in the previous section, yields different type of partial derivatives of a function.

1.1. Definitions. Consider a real valued function $z = f(x, y)$ defined on $E \subset \mathbb{R}^2$ such that E contains a neighbourhood of $(a, b) \in \mathbb{R}^2$. Let Δa be a change in a. If the limit,

$$\lim_{\Delta a \to 0} \frac{f(a + \Delta a, b) - f(a, b)}{\Delta a}$$

exists, then it is called the *partial derivative of f with respect to x at (a, b)* and is denoted by $\frac{\partial f}{\partial x}\Big|_{(a,b)}$ or $f_x(a, b)$ or $z_x(a, b)$. Similarly, let Δb be a change in b. If the limit,

$$\lim_{\Delta b \to 0} \frac{f(a, b + \Delta b) - f(a, b)}{\Delta b}$$

exists, then it is called the *partial derivative of f with respect to y at (a, b)* and is denoted by $\frac{\partial f}{\partial y}\Big|_{(a,b)}$ or $f_y(a, b)$ or $z_y(a, b)$.

Notations. If the partial derivatives f_x and f_y exist at each point of E, then they are also the real valued functions on E. Further, we can obtain the partial derivatives of these functions, if they are differentiable. In these cases, we fix up the following notations.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right),$$
$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right); \text{ and } f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

The notations of derivatives of order greater than two should be clear from the above pattern.

1.2. Remark. As we have seen in the above example, in general, f_{xy} and f_{yx} need not be equal, even if they exist. The following proposition gives a sufficient condition for them to be equal. We accept it without proof. However, we shall be dealing only with the functions f for which these two are equal.

1.3. Proposition. *Consider a real valued function $z = f(x, y)$ defined on $E \subset \mathbb{R}^2$ such that E contains a neighbourhood of $(a, b) \in \mathbb{R}^2$. If f_{xy} and f_{yx} exist and are continuous, then $f_{xy} = f_{yx}$.*

Throughout this chapter our blanket assumption will be that the operation of taking partial derivatives is commutative. That is, for our function f of two variables, $f_{xy} = f_{yx}$. In general, we may assume that the second derivatives of functions exists and are continuous, so that, the Proposition 1.3 ensures our requirement.

1.4. Example. For $u = x^3 - 3xy^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Also prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

SOLUTION. Here $u = x^3 - 3xy^2$. Hence,

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2; \quad \frac{\partial u}{\partial y} = -6xy; \quad \frac{\partial^2 u}{\partial x \partial y} = -6y = \frac{\partial^2 u}{\partial y \partial x}.$$

$$\frac{\partial^2 u}{\partial x^2} = 6x; \quad \frac{\partial^2 u}{\partial y^2} = -6x.$$

Hence,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

□

2. Homogeneous functions

Let us observe the following expressions carefully.

(1) $f_1(x, y) = x^2y^4 - x^3y^3 + xy^5$.
(2) $f_2(x, y) = x^4y^4 - x^5y^3 + x^6y^2$.

The combined degree of x and y in each term of the first expression is 6 and that in the second expression is 8. Can we determine whether the combined degree of x and y in each term of the expression $\frac{x}{x^4+y^4}$ is same or not? It seems difficult to determine. Let us develop the following tests.

Test 1: Let us take $t = \frac{y}{x}$. Then

$$x^2y^4 - x^3y^3 + xy^5 = x^6(t^4 - t^3 + t^5) = x^6 f(t)$$

and

$$x^4y^4 - x^5y^3 + x^6y^2 = x^8(t^4 - t^3 + t^2) = x^8 g(t),$$

where f and g are functions of one variable t.

Test 2: Now, let us replace x by tx and y by ty. Then

$$f_1(tx, ty) = (tx)^2(ty)^4 - (tx)^3(ty)^3 + (tx)(ty)^5 = t^6 f_1(x, y)$$

and

$$f_2(tx, ty) = (tx)^4(ty)^4 - (tx)^5(ty)^3 + (tx)^6(ty)^2 = t^8 f_2(x, y).$$

2.1. Definitions. A function $z = f(x, y)$ is said to be a *homogeneous function of degree r*, if $f(tx, ty) = t^r f(x, y)$ for some real number r. Otherwise, f is said to be a *nonhomogeneous function*.

2.2. Example. Let $f : \mathbb{R}^2 \setminus \{(x, y) : y = -x\} \to \mathbb{R}$ defined by $f(x, y) = \frac{x-y}{x+y}$. Then prove that f is a homogeneous function of degree 0 and f_x and f_y exist at each point of the domain.

SOLUTION. Clearly, $f(tx, ty) = f(x, y) = t^0 f(x, y)$. Thus f is a homogeneous function of degree 0. Now for any $(x, y) \in \mathbb{R}^2$ with $x + y \neq 0$, we have,

$$f_x(x, y) = \frac{(x + y)(1) - (x - y)(1)}{(x + y)^2} = \frac{2y}{(x + y)^2}$$

and

$$f_y(x, y) = \frac{(x + y)(-1) - (x - y)(1)}{(x + y)^2} = \frac{-2x}{(x + y)^2}.$$

□

2.3. Example. $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ defined by $f(x, y) = \frac{\sqrt[5]{x} - \sqrt[5]{y}}{x^3 + y^3}$ is a homogeneous function of degree $-\frac{14}{5}$.

2.4. Theorem (Euler's Theorem)**.** *State and prove Euler's Theorem*
Statement : Let $z = f(x, y)$ be a real valued function defined on $E \subset \mathbb{R}^2$. Suppose that f is a homogeneous function of degree n. If f_x and f_y exist on E, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz. \tag{2.4.1}$$

PROOF. Since $z = f(x, y)$ is a homogeneous function of x, y of degree n, we can write

$$z = f(x, y) = x^n g\left(\frac{y}{x}\right). \tag{2.4.2}$$

Differentiating (2.4.2) partially with respect to x, we get,

$$\frac{\partial z}{\partial x} = nx^{n-1} g\left(\frac{y}{x}\right) + x^n g'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right).$$

Hence,

$$x\frac{\partial z}{\partial x} = nx^n g\left(\frac{y}{x}\right) - x^{n-1} y g'\left(\frac{y}{x}\right). \tag{2.4.3}$$

Similarly, differentiating (2.4.2) partially with respect to y, we get,

$$\frac{\partial z}{\partial y} = x^n g'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1} g'\left(\frac{y}{x}\right).$$

Hence,

$$y\frac{\partial z}{\partial y} = yx^{n-1} g'\left(\frac{y}{x}\right). \tag{2.4.4}$$

Adding (2.4.3) and (2.4.4) we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n g\left(\frac{y}{x}\right) = nz.$$

This completes the proof. □

We note that the converse of Euler's Theorem also holds. That is, if a function $z = f(x, y)$ satisfies (2.4.1), on a certain domain, then it must be homogeneous on that domain.

2.5. Remark. Now onwards we shall not mention the domain of the functions under discussion. Also, whenever we use the derivatives of functions under discussion, we assume them to be sufficiently many times differentiable.

2.6. Corollary. *Let $z = f(x, y)$ be a real valued function defined on $E \subset \mathbb{R}^2$. Suppose that f is a homogeneous function of degree n and that all the second order partial derivatives of f exist and are continuous. Then prove that*

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

Proof. Since $z = f(x, y)$ is a homogeneous function of x, y of degree n, by Euler's Theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz. \tag{2.6.1}$$

Differentiating (2.6.1) partially with respect to x, we have,

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x \partial y} = n\frac{\partial z}{\partial x},$$

which, on multiplication by x, gives

$$x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} + xy\frac{\partial^2 z}{\partial x \partial y} = nx\frac{\partial z}{\partial x}.$$

Hence,

$$x^2\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial x \partial y} = (n-1)x\frac{\partial z}{\partial x}. \tag{2.6.2}$$

Similarly, differentiating (2.6.1) partially with respect to y and then multiplying the result by y, we get,

$$y^2\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial^2 z}{\partial y \partial x} = (n-1)y\frac{\partial z}{\partial y}.$$

Since $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$, we get,

$$y^2\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial^2 z}{\partial x \partial y} = (n-1)y\frac{\partial z}{\partial y}. \tag{2.6.3}$$

By adding (2.6.2) and (2.6.3) we have,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}) = n(n-1)z.$$

This completes the proof. □

2.7. Corollary. *Let $u = u(x, y)$ be a nonhomogeneous real valued function defined on $E \subset \mathbb{R}^2$ and $z = \varphi(u)$ be homogeneous function of degree n. Then prove that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{\varphi(u)}{\varphi'(u)},$$

provided $\varphi'(u) \neq 0$ for any $(x, y) \in E$.

PROOF. Since $z = \varphi(u)$ is a homogeneous function of x, y of degree n, by Euler's Theorem we have,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = n\varphi(u)$$

$$\Rightarrow x\left(\varphi'(u)\frac{\partial u}{\partial x}\right) + y\left(\varphi'(u)\frac{\partial u}{\partial y}\right) = n\varphi(u) \Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = n\frac{\varphi(u)}{\varphi'(u)}.$$

□

2.8. Corollary. *(Only statement)Let $u = u(x, y)$ be a nonhomogeneous real valued function defined on $E \subset \mathbb{R}^2$ and $z = \varphi(u)$ be homogeneous function of degree n. Then prove that*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \psi(u)[\psi'(u) - 1],$$

where $\psi(u) = n\frac{\varphi(u)}{\varphi'(u)}$, provided $\varphi'(u) \neq 0$ for any $(x, y) \in E$.

2.9. Example. For the following functions, verify Euler's Theorem and find $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}$.

(1) $z = x^n \log\left(\frac{y}{x}\right)$.

(2) $z = \sin^{-1}(\frac{x}{y}) + \tan^{-1}(\frac{y}{x})$.

SOLUTION. (1) Clearly, z is a homogeneous function of degree n.

$$\frac{\partial z}{\partial x} = nx^{n-1}\log\left(\frac{y}{x}\right) + x^n\frac{x}{y}\left(-\frac{y}{x^2}\right) = nx^{n-1}\log\left(\frac{y}{x}\right) - x^{n-1}$$

$$\Rightarrow x\frac{\partial z}{\partial x} = nx^n\log\left(\frac{y}{x}\right) - x^n.$$

Also,

$$\frac{\partial z}{\partial y} = x^n\left(\frac{x}{y}\right)\left(\frac{1}{x}\right) = \frac{x^n}{y} \Rightarrow y\frac{\partial z}{\partial y} = x^n.$$

Hence,

$$\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n\log\left(\frac{y}{x}\right) = nz.$$

Thus Euler's Theorem is verified.

By the Corollary 2.6,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

(2) Replacing x by tx and y by ty, $f(tx, ty) = \sin^{-1}(\frac{x}{y}) + \tan^{-1}(\frac{y}{x}) = t^0 f(x, y)$. Thus $z = f(x, y)$ is a homogeneous function of degree 0. Now,

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

$$\Rightarrow x\frac{\partial z}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2}.$$

Also,

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{-x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$\Rightarrow y\frac{\partial z}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}.$$

Hence,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

Thus Euler's Theorem is verified.

By the Corollary 2.6,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z = 0,$$

as $n = 0$. □

2.10. Example. If $u = \sin^{-1}(\frac{x^2y^2}{x+y})$, then prove the following.

(1) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$.

(2) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 3\tan u(3\sec^2 u - 1)$.

Solution. Here $u = \sin^{-1}(\frac{x^2y^2}{x+y})$ is not a homogeneous function of x, y. Writing the given equation differently, we have $\sin u = \frac{x^2y^2}{x+y}$. Let $z = \varphi(u) = \sin u$. Then $z = \frac{x^2y^2}{x+y}$, which is homogeneous of degree 3. Hence by Corollary 2.7, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\frac{\varphi(u)}{\varphi'(u)} = 3\frac{\sin u}{\cos u} = 3\tan u$, which proves (1). Also, by Corollary 2.8, we have,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 3\tan u[3\sec^2 u - 1].$$

□

3. Theorem on total differentials

Throughout this section we consider only those functions of two variables that admit continuous partial derivatives on their domain of definition. That is, if we are discussing about a function $z = f(x, y)$, then f_x, f_y exist and are continuous on the domain of f.

3.1. Theorem. *(Only statement) Let $z = f(x, y)$ be defined on E. Then*

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

4. Differentiation of composite functions

In this section we shall study the differentiation of composite functions. Let $z = f(x, y)$ be function defined on $E \subset \mathbb{R}^2$. In turn one can have $x = \phi(t)$ and $y = \psi(t)$, $t \in F \subset \mathbb{R}$. This makes f a function of one independent variable t. That is,

$$t \in F \mapsto (\phi(t), \psi(t)) \in E \mapsto f(\phi(t), \psi(t)).$$

The following theorem describes the differentiation of f with respect to t in this situation.

4.1. Theorem. *(Only statement) Let $z = f(x,y)$ be function defined on $E \subset \mathbb{R}^2$ and $x = \phi(t)$, $y = \psi(t)$, $t \in F \subset \mathbb{R}$. Then prove that $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.*

To extend Theorem 4.1 for functions of three variables, let $u = f(x,y,z)$ be a function of three variables with $x = x(t)$, $y = y(t)$ and $z = z(t)$. Then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

5. Change of variables

Like the composite functions we can also consider the following situation. Let $z = f(x,y)$ be function defined on $E \subset \mathbb{R}^2$ and let there be another domain $F \subset \mathbb{R}^2$ such that for each $(x,y) \in E$, $x = \phi(u,v)$, $y = \psi(u,v)$, $(u,v) \in F \subset \mathbb{R}^2$. This is nothing but the change of variable. In this case, the following theorem describes the partial derivatives of f with respect to u and v.

Now we prove Euler's Theorem for three variables. The homogeneous functions of more than two variables are defined as in Definition 2.1. More explicitly, a function $H = f(x_1, x_2, \ldots, x_n)$ of n variables is called *homogeneous* if there exists $r \in \mathbb{R}$ such that for $f(tx_1, tx_2, \ldots, tx_n) = t^r f(x_1, x_2, \ldots, x_n)$ for all $t \in \mathbb{R}$. In this case, the degree of homogeneity of H is r.

5.1. Theorem (Euler's Theorem for Three variables)**.** *Let $H = f(x,y,z)$ be a real valued homogeneous function of three variables x,y,z of degree n defined on $E \subset \mathbb{R}^3$. If f_x, f_y, f_z exist on E, then prove that*

$$x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} + z\frac{\partial H}{\partial z} = nH. \tag{5.1.1}$$

Proof. Since $H = f(x,y,z)$ is homogeneous function of degree n,

$$H = x^n \varphi\left(\frac{y}{x}, \frac{z}{x}\right) = x^n \varphi(u,v),$$

where $u = \frac{y}{x}$ and $v = \frac{z}{x}$. Hence,

$$\begin{aligned}
\frac{\partial H}{\partial x} &= nx^{n-1}\varphi(u,v) + x^n\left[\frac{\partial \varphi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v}\frac{\partial v}{\partial x}\right]\\
&= nx^{n-1}\varphi(u,v) + x^n\left[-\frac{y}{x^2}\frac{\partial \varphi}{\partial u} - \frac{z}{x^2}\frac{\partial \varphi}{\partial v}\right]\\
&= nx^{n-1}\varphi(u,v) - x^{n-2}y\frac{\partial \varphi}{\partial u} - x^{n-2}z\frac{\partial \varphi}{\partial v}\\
\Rightarrow x\frac{\partial H}{\partial x} &= nx^n\varphi(u,v) - x^{n-1}y\frac{\partial \varphi}{\partial u} - x^{n-1}z\frac{\partial \varphi}{\partial v}. \qquad (5.1.2)
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{\partial H}{\partial y} &= x^n\left[\frac{\partial \varphi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v}\frac{\partial v}{\partial y}\right] = x^n\left[\frac{1}{x}\frac{\partial \varphi}{\partial u} + 0\frac{\partial \varphi}{\partial v}\right] = x^{n-1}\frac{\partial \varphi}{\partial u}\\
\Rightarrow y\frac{\partial H}{\partial y} &= x^{n-1}y\frac{\partial \varphi}{\partial u}. \qquad (5.1.3)
\end{aligned}$$

Similarly,

$$z\frac{\partial H}{\partial z} = x^{n-1} z\frac{\partial \varphi}{\partial v}. \tag{5.1.4}$$

Adding (5.1.2), (5.1.3) and (5.1.4) we have,

$$x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} + z\frac{\partial H}{\partial z} = nx^n\varphi(u,v) = nH.$$

This completes the proof. □

As noted in case of the functions of two variables, here also we recall that the converse of Euler's Theorem also holds. That is, if a function $z = f(x, y)$ satisfies (5.1.1), on a certain domain, then it must be homogeneous on that domain.

5.2. Example. Find $\frac{dz}{dt}$ when $z = \sin^{-1}(x - y)$, $x = 3t,\ y = 4t^3$. Also verify by the direct substitution.

SOLUTION.

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= \frac{1}{\sqrt{1-(x-y)^2}} \cdot 3 - \frac{1}{\sqrt{1-(x-y)^2}} \cdot 12t^2 \\
&= \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}} \\
&= \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}} \\
&= \frac{3(1-4t^2)}{\sqrt{(1-3t+4t^3)(1+3t-4t^3)}} \\
&= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}.
\end{aligned}$$

On the other hand, verifying directly by putting the values of x and y in z, we have

$$\begin{aligned}
& z = \sin^{-1}(3t - 4t^3) \\
\Rightarrow & \frac{dz}{dt} = \frac{(3-12t^2)}{\sqrt{1-(3t-4t^3)^2}} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}} = \frac{3}{\sqrt{1-t^2}}.
\end{aligned}$$

□

5.3. Example. If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, then prove that

$$\left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2 = \left[\frac{\partial z}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial z}{\partial \theta}\right]^2.$$

SOLUTION. Here x, y are functions of r, θ. Hence z is a composite function of r, θ. Thus,

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}$$

$$\Rightarrow \left[\frac{\partial z}{\partial r}\right]^2 = \cos^2\theta\left[\frac{\partial z}{\partial x}\right]^2 + 2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \sin^2\theta\left[\frac{\partial z}{\partial y}\right]^2. \tag{5.3.1}$$

Also,

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}$$

$$\Rightarrow \left[\frac{\partial z}{\partial \theta}\right]^2 = r^2\sin^2\theta\left[\frac{\partial z}{\partial x}\right]^2 - 2r^2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + r^2\cos^2\theta\left[\frac{\partial z}{\partial y}\right]^2$$

$$\Rightarrow \frac{1}{r^2}\left[\frac{\partial z}{\partial \theta}\right]^2 = \sin^2\theta\left[\frac{\partial z}{\partial x}\right]^2 - 2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \cos^2\theta\left[\frac{\partial z}{\partial y}\right]^2. \tag{5.3.2}$$

Adding (5.3.1) and (5.3.2) we get,

$$\left[\frac{\partial z}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial z}{\partial \theta}\right]^2 = \left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2.$$

□

5.4. Example. If $H = f(2x-3y, 3y-4z, 4z-2x)$, then prove that

$$\frac{1}{2}\frac{\partial H}{\partial x} + \frac{1}{3}\frac{\partial H}{\partial y} + \frac{1}{4}\frac{\partial H}{\partial z} = 0.$$

Solution. Let $u = 2x-3y$, $v = 3y-4z$, $w = 4z-2x$. Then $H = f(u,v,w)$. Hence H is a composite function of x, y, z. Therefore,

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial H}{\partial w}\frac{\partial w}{\partial x} = 2\frac{\partial H}{\partial u} + 0\frac{\partial H}{\partial v} - 2\frac{\partial H}{\partial w} = 2\frac{\partial H}{\partial u} - 2\frac{\partial H}{\partial w}. \tag{5.4.1}$$

Also,

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial H}{\partial w}\frac{\partial w}{\partial y} = -3\frac{\partial H}{\partial u} + 3\frac{\partial H}{\partial v} + 0\frac{\partial H}{\partial w} = -3\frac{\partial H}{\partial u} + 3\frac{\partial H}{\partial v}. \tag{5.4.2}$$

Finally,

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial H}{\partial w}\frac{\partial w}{\partial z} = 0\frac{\partial H}{\partial u} - 4\frac{\partial H}{\partial v} + 4\frac{\partial H}{\partial w} = -4\frac{\partial H}{\partial v} + 4\frac{\partial H}{\partial w}. \tag{5.4.3}$$

Hence,

$$\frac{1}{2}\frac{\partial H}{\partial x} + \frac{1}{3}\frac{\partial H}{\partial y} + \frac{1}{4}\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u} - \frac{\partial H}{\partial w} - \frac{\partial H}{\partial u} + \frac{\partial H}{\partial v} - \frac{\partial H}{\partial v} + \frac{\partial H}{\partial w} = 0.$$

□

5.5. Example. If $z = f(x,y)$ and $u = e^x\cos y$, $v = e^x\sin y$. Then prove that $\frac{\partial f}{\partial x} = u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}$.

Solution. $u = e^x\cos y$, $v = e^x\sin y$. Hence,

$$u^2 + v^2 = e^{2x} \Rightarrow e^x = \sqrt{u^2+v^2} \Rightarrow x = \frac{1}{2}\log(u^2+v^2).$$

Also,

$$\frac{v}{u} = \tan y \Rightarrow y = \tan^{-1}(\tfrac{v}{u}).$$

Thus x, y are functions of u, v, and so, z is a composite function of u, v. Now,

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}\left[\frac{u}{u^2+v^2}\right] + \frac{\partial f}{\partial y}\left[\frac{-v}{u^2+v^2}\right]$$

$$\Rightarrow u\frac{\partial f}{\partial u} = \left[\frac{u^2}{u^2+v^2}\right]\frac{\partial f}{\partial x} - \left[\frac{uv}{u^2+v^2}\right]\frac{\partial f}{\partial y}. \tag{5.5.1}$$

Similarly,

$$v\frac{\partial f}{\partial v} = \left[\frac{v^2}{u^2+v^2}\right]\frac{\partial f}{\partial x} + \left[\frac{uv}{u^2+v^2}\right]\frac{\partial f}{\partial y}. \tag{5.5.2}$$

Adding (5.5.1) and (5.5.2) we get, $u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}$. □

6. Differentiation of implicit functions

Many a times we are given an expression $f(x,y) = c$, where $c \in \mathbb{R}$ is a constant. Note here that, x and y are associated by a rule however we may not be able to write y as a function of x. In this case, we say that y is a function of x, implicitly described by $f(x,y) = c$ or y is an implicit function of x. We obtain the method of calculating $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ using the tools of partial derivatives.

6.1. Theorem. *Let a function y of x be implicitly described by $f(x,y) = c$. Then prove that*

(1) $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$.

(2) $\dfrac{d^2y}{dx^2} = -\dfrac{f_{xx}(f_y)^2 - 2f_{xy}f_xf_y + f_{yy}(f_x)^2}{(f_y)^3}$.

PROOF. We know that f is a function of x and y. Also, y is an implicit function of x. So, f is a composite function of x. Hence, differentiating the equation $f(x,y) = c$ with respect to x, we get,

$$\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}.$$

This proves (1).

Now we prove (2).

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{d}{dx}\left(\frac{-f_x}{f_y}\right) \\
&= -\frac{f_y\frac{d}{dx}(f_x) - f_x\frac{d}{dx}(f_y)}{(f_y)^2} \\
&= -\frac{f_y\left(\frac{\partial}{\partial x}(f_x) + \frac{\partial}{\partial y}(f_x)\frac{dy}{dx}\right) - f_x\left(\frac{\partial}{\partial x}(f_y) + \frac{\partial}{\partial y}(f_y)\frac{dy}{dx}\right)}{(f_y)^2} \\
&= -\frac{f_y\left(f_{xx} + f_{xy}\left(-\frac{f_x}{f_y}\right)\right) - f_x\left(f_{yx} + f_{yy}\left(-\frac{f_x}{f_y}\right)\right)}{(f_y)^2} \\
&= -\frac{f_{xx}(f_y)^2 - f_yf_xf_{xy} - f_xf_yf_{yx} + f_{yy}(f_x)^2}{(f_y)^3}
\end{aligned}$$

$$= -\frac{f_{xx}(f_y)^2 - 2f_x f_y f_{xy} + f_{yy}(f_x)^2}{(f_y)^3}.$$

□

6.2. Example. Find $\frac{dy}{dx}$ when

(1) $x\sin(x-y) - (x+y) = 0$. (2) $x^y = y^x$.

PROOF. (1) Let $f(x,y) = x\sin(x-y)-(x+y)$. Since $f(x,y) = 0$, by the previous theorem, we have,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{x\cos(x-y) + \sin(x-y) - 1}{x\cos(x-y)(-1) - 1}$$
$$= \frac{x\cos(x-y) + \sin(x-y) - 1}{x\cos(x-y) + 1}.$$

(2) Let $f(x,y) = x^y - y^x$. Since $f(x,y) = 0$, by the previous theorem, we have,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{yx^{y-1} - y^x \log y}{x^y \log x - xy^{x-1}} = \frac{y^x \log y - yx^{y-1}}{x^y \log x - xy^{x-1}}.$$

□

6.3. Example. If $z = xyf(\frac{y}{x})$ and z is constant, then show that

$$\frac{f'(\frac{y}{x})}{f(\frac{y}{x})} = \frac{x[y + x\frac{dy}{dx}]}{y[y - x\frac{dy}{dx}]}.$$

SOLUTION. Let $F(x,y) = xyf(\frac{y}{x})$. Then $F(x,y) = z$, z is constant. Thus y is an implicit function of x. So,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0. \qquad (6.3.1)$$

Now differentiating $F(x,y)$ with respect to x, we get,

$$\frac{\partial F}{\partial x} = yf(\tfrac{y}{x}) + xyf'(\tfrac{y}{x})\left(-\frac{y}{x^2}\right) = yf(\tfrac{y}{x}) - \frac{y^2}{x}f'(\tfrac{y}{x}) = \frac{y}{x}\left[xf(\tfrac{y}{x}) - yf'(\tfrac{y}{x})\right].$$

Similarly,

$$\frac{\partial F}{\partial y} = xf(\tfrac{y}{x}) + xyf'(\tfrac{y}{x})(\tfrac{1}{x}) = xf(\tfrac{y}{x}) + yf'(\tfrac{y}{x}).$$

Putting these values in (6.3.1), we have,

$$\frac{y}{x}\left[xf(\tfrac{y}{x}) - yf'(\tfrac{y}{x})\right] + xf(\tfrac{y}{x}) + yf'(\tfrac{y}{x})\frac{dy}{dx} = 0$$
$$\Rightarrow \left[y + x\frac{dy}{dx}\right] f(\tfrac{y}{x}) = \frac{y}{x}\left[y - x\frac{dy}{dx}\right] f'(\tfrac{y}{x})$$
$$\Rightarrow \frac{f'(\frac{y}{x})}{f(\frac{y}{x})} = \frac{x}{y}\left[\frac{y + x\frac{dy}{dx}}{y - x\frac{dy}{dx}}\right].$$

□

6.4. Example. If A, B and C are angles of a ΔABC such that $\sin^2 A + \sin^2 B + \sin^2 C = K$, a constant, then prove that $\frac{dB}{dC} = \frac{\tan C - \tan A}{\tan A - \tan B}$.

SOLUTION. Clearly, $A + B + C = \pi$. So, $A = \pi - (B + C)$. Therefore, $\sin A = \sin(B + C)$. Let $f(B, C) = \sin^2(B + C) + \sin^2 B + \sin^2 C - K$. Hence $f(B, C) = 0$, *i.e.*, B is an implicit function of C. So, $\frac{dB}{dC} = -\frac{f_C}{f_B}$. Also,

$$\begin{aligned} f_B &= \frac{\partial f}{\partial B} \\ &= 2\sin(B + C)\cos(B + C) + 2\sin B \cos B \\ &= \sin 2(B + C) + \sin 2B \\ &= \sin(2\pi - 2A) + \sin 2B \\ &= -\sin 2A + \sin 2B \\ &= 2\cos(B + A)\sin(B - A) \\ &= 2\cos(\pi - C)\sin(B - A) \\ &= -2\cos C \sin(B - A) \\ &= 2\cos C \sin(A - B). \end{aligned}$$

Similarly, we get,

$$f_C = 2\cos B \sin(A - C).$$

Hence,

$$\begin{aligned} \frac{dB}{dC} &= -\frac{\cos B \sin(A - C)}{\cos C \sin(A - B)} \\ &= -\frac{\cos B(\sin A \cos C - \cos A \sin C)}{\cos C(\sin A \cos B - \cos A \sin B))} \\ &= -\frac{\sin A \cos B \cos C - \cos A \cos B \sin C}{\sin A \cos B \cos C - \cos A \sin B \cos C}. \end{aligned}$$

Dividing by $\cos A \cos B \cos C$, we get,
$\frac{dB}{dC} = -\frac{\tan A - \tan C}{\tan A - \tan B} = \frac{\tan C - \tan A}{\tan A - \tan B}$.

□

♣♣♣♣♣♣♣♣♣