Graph Theory [US06CMTH05]

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Unit : 3

CONTENTS

Spanning Trees , Fundamental circuits , Finding all spanning trees of a graph , Cut-sets and their properties , All cut-sets in a graph , Fundamental circuits and cut sets , Connectivity and separability , Network flows.

3.1. Definition. Spanning Tree

A subgraph T of a connected graph G is called a spanning tree if it is a tree containing all the vertices of G.In the figure : 1 a spanning tree is indicated by thick edges.

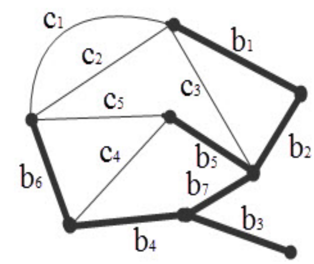


Figure 1. A spanning tree indicated by thick edges

3.2. Definition. Branch

Branch :
An edge in a spanning tree T is called a **branch** of the spanning tree T. In the figure 1 the edges $b_1, b_2, \ldots, b_7$ are branches of the tree indicated by thick edges.

3.3. Definition. Chord

Chord :
If an edge of a connected graph G is not a branch of a spanning tree T then the edge is called a chord of the spanning tree T. In the figure 1 the edges c_1, c_2, c_3, c_4 and c_7 are chords of the tree indicated by thick edges.

3.4. Definition. Forest

A collection of trees in a graph G is called a **forest** in the graph.

3.5. Definition. Spanning Forest

A forest that contains every vertex of a graph G such that two vertices are in the same tree of the forest when there is a path in G between these two vertices. In other words, a spanning forest of a graph G is a collection of exactly one spanning tree from each of its connected components.

3.6. Definition. Rank

Rank
If in a graph G there are total n vertices and k components then the rank, generally denoted by r, is defined as

$$r = n - k$$

3.7. Definition. Nullity

<u>Nullity</u>
If in a graph G there are total n vertices, k components and e edges then the nullity of G, generally denoted by μ, is defined as

$$\mu = e - n + k$$

3.8. Definition. Fundamental Circuit

<u>Fundamental Circuit</u>
Let T be a spanning tree in a connected graph G. When a chord is added to a spanning tree T then it forms exactly one circuit. Such a circuit is called a fundamental circuit.

3.9. THEOREM. Prove that every connected graph has at least one spanning tree.

Proof : If a connected graph G has no circuit, then G itself is a tree through all its vertices. Therefore, G is its own spanning tree.

Figure 2

Now, if G has one or more circuits then choose any circuit and delete an edge from the circuit. In the chosen circuit there is atleast one more path that connects the end vertices of the deleted edge. Therefore the deletion of the edge will still leave the graph connected.

If there are more circuits after the deletion then again choose any circuit from the graph and delete any one of its edge. Again the resultant graph will remain connected.

Continue the operation of deletion repeatedly so that all the circuits are 'broken' and the resultant subgraph is connected and circuit-free that contains all the vertices of G.

Hence, at the end of the above prodecure we always obtain a spnning tree.

Thus, every connected graph has atleast one spanning tree.

3.10. Example. Describe a method to find all spanning tree of a graph.

Solution : Let G be a connected graph. If G is a tree then G itself is one and only one spanning tree of G.
Now, as shown in the figure 3, consider a connected graph G. It is not a tree because it has atleat one circuit. Let T_1 be a spanning tree

of G that consists of the branches a, b, c, d.

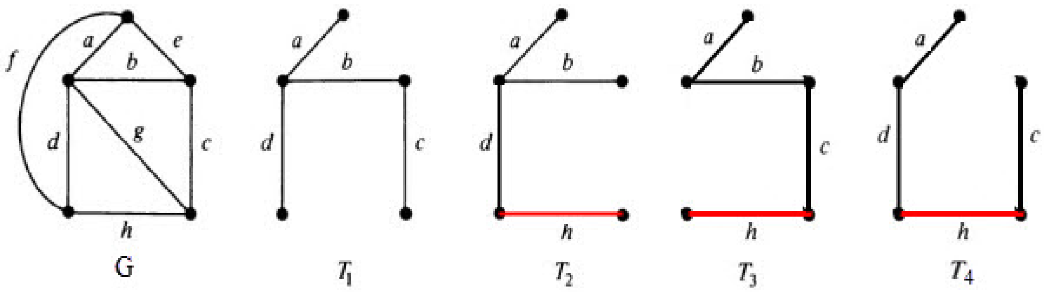


FIGURE 3. Finding a spanning tree

Add a chord, say h, to the tree which will form a fundamental circuit through b, c, h, d. Removal of the branch c of T_1 from the fundamental circuit b, c, h, d will break the circuit and create another spanning tree, say T_2.

Instead of deleting c, we can delete d or b and obtain two more different spanning trees namely a, b, c, h and a, c, h, d. This process generates all possible trees corresponding to the chord h and associated fundamental circuit.

We restart with the initial tree T_1 and repeat the process, that we followed with the chord h, using another chord e or f or g and obtain all possible different spanning trees corresponding to each chord addition to T_1.

Thus, we can obtain all possible spanning trees of a connected graph.

3.11. DEFINITION. Cut-Set

In a connected graph G, a set of edges whose removal from

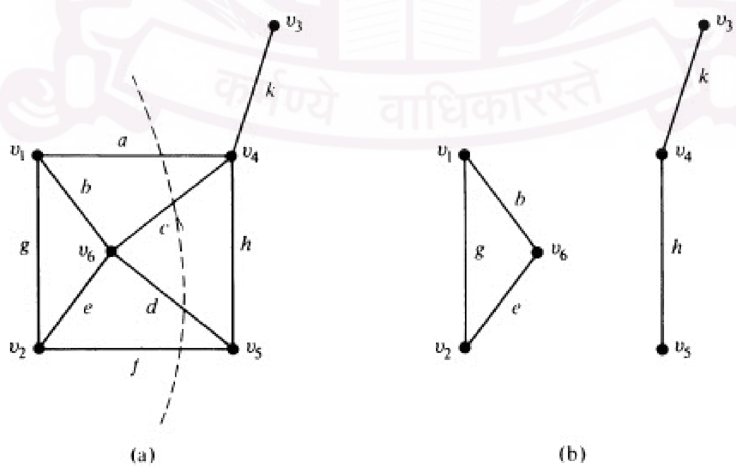


FIGURE 4. Cut Set

the graph leaves the graph disconnected, provided removal of no proper subsets of these edges disconnets G. is called a **cut-set** of the graph.

In the graph in Figure : 4 the set of edges $\{a, c, d, f\}$ is a cut set of the graph.

3.12. THEOREM. Prove that every cut-set in a connected graph G must contain at least one branch of every spanning tree.

Proof : Let G be a connected graph and S be a cut-set of G.
If possible, suppose T is spannig tree of G which has no edge included in the cut-set S. Therefore T is completely contained in $G - S$.

As T is a spanning tree and spans through all the vertices of G, the sub-graph $G - S$ remains connected.
But, that is not possible as removal of a cut-set must leave the graph disconnected.

Therefore, our supposion is wrong. Hence, every cut-set in a connected graph G must contain at least one branch of every spanning tree.

3.13. THEOREM. Prove that in a connected graph G any minimal set of edges containing at least one branch of every spanning tree of G is a cut-set.

Proof : Let G be a connected graph and Q be a minimal set of edges containing atleast one branch of every spanning tree of G.

Now, $G - Q$ is a subgraph of G from which atleast one branch of every spanning tree is missing.

As $G - Q$ cannot contain any spanning tree of G completely, it must be disconnected.

Since, Q is a minimal set of edges with this property, any edge e returned from G to $G - Q$ will create atleast one spanning tree.
Therefore, $G - Q + e$ will be a connected graph.

Thus, Q is a minimal set of edges whose removal from G disconnects G.

Hence, Q is a cut-set of G.

3.14. THEOREM. Prove that every circuit has an even number of edges in common with any cut-set.

Proof : Consider a cut-set S in a connected graph G .
Let the removal of S partition the vertices of G into two disjoint subsets V_1 and V_2.
Let τ be a circuit in G. If all the vertices of τ lie entirely within one of the vertex sets V_1 or V_2, then all the edges of τ are different from those of S. Therefore in that case the number of edges common to S and τ is zero; that is,

$$N(S \cap \tau) = 0$$

, an even number.
Now, if some vertices in τ are in V_1 and some are in V_2, we need to traverse back and forth between the sets V_1 and V_2 as we traverse the circuit.

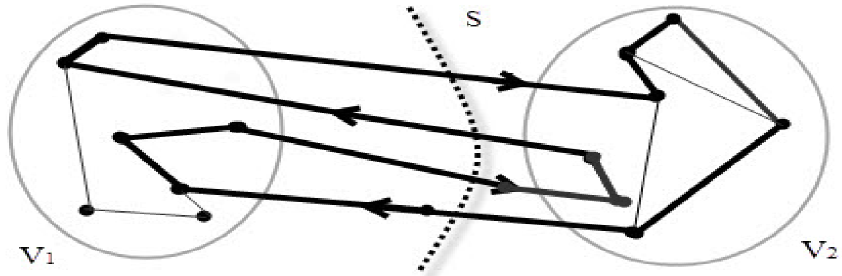


FIGURE 5. Circuit (thick lines) and a cut-set in a graph G

Because of the closed nature of a circuit, the number of edges we traverse between V_1 and V_2 must be even.

Also every edge in S has one end in V_1 and the other in V_2, and no other edge in G has this property of separating sets V_1 and V_2, the number of edges common to S and τ is even.

Thus, in any case every circuit has an even number of edges in common with any cut-set.

3.15. DEFINITION. Fundamental cut-set

Let T be a spanning tree of a connected graph G. Then a cutset formed by exacyly one branch, say b, of T and possibly some more chords of T is called a Fundamental cut-set of G relative to the spanning tree T.
In the figure : 6 $\{d, e, f\}$ is a fundamental cut-set with d as one of

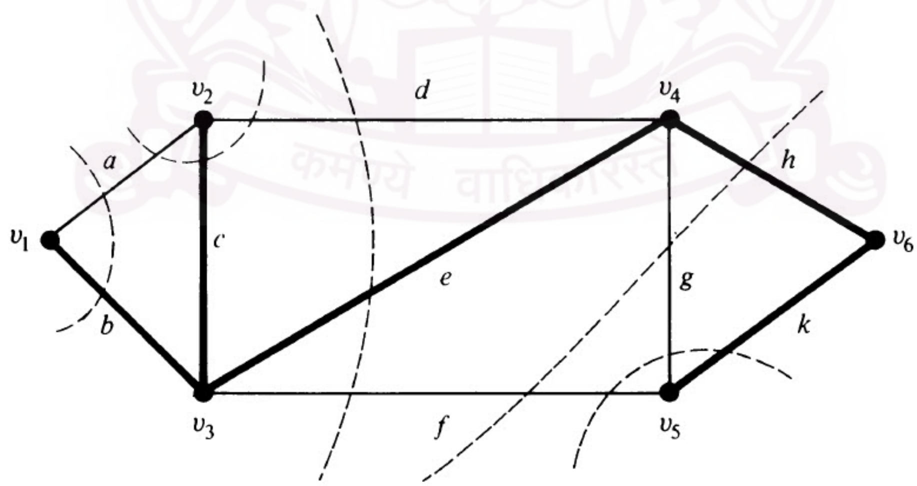
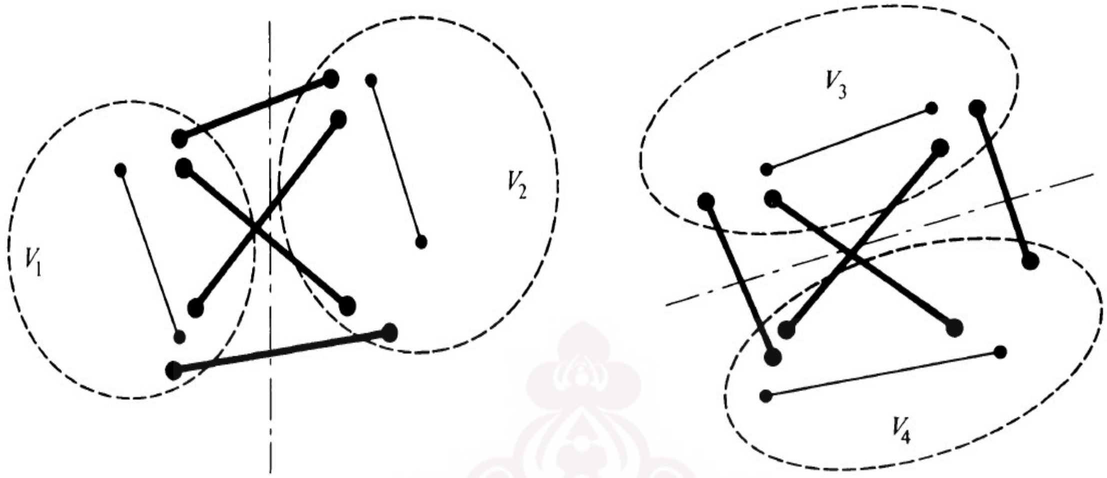


FIGURE 6. Fundamental Cut-Set

the branch of a tree (thick lines) and remaining edges d and f as chords of the corresponding tree.

3.16. THEOREM. Prove that the ring sum of any two cut-sets is either a cut-set or an edge disjoint union of cut-sets.

Proof : Let G be a connected graph with vertex set V.
Suppose S_1 and S_2 are two cut-sets in G.



(A) Partition by S_1 (B) Partition by S_2

FIGURE 7. Cutset partitions

If V_1 and V_2 form the unique and disjoint partitioning of V corresponding to S_1 (Figure 7a) and V_3 and V_4 is the unique and disjoint partitioning of V corresponding to S_2 (figure 7b) then clearly,

$$V_1 \cup V_2 = V \quad \text{and} \quad V_1 \cap V_2 = \emptyset$$
$$V_3 \cup V_4 = V \quad \text{and} \quad V_3 \cap V_4 = \emptyset$$

Now let

$$V_5 = (V_1 \cap V_4) \cup (V_2 \cap V_3)$$

and

$$V_6 = (V_1 \cap V_3) \cup (V_2 \cap V_4)$$

From the Figure 8 it can be seen that

$$V_5 = V_1 \oplus V3 \quad \text{and} \quad V6 = V_2 \oplus V3$$

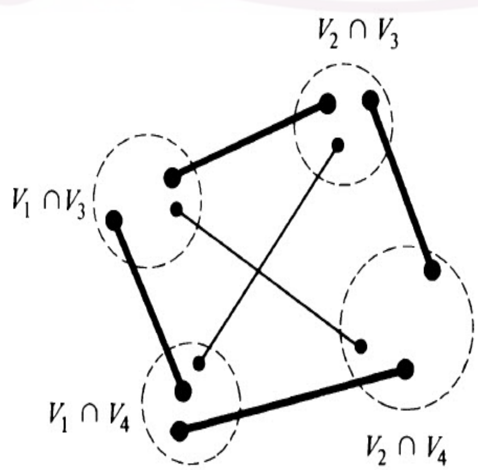


FIGURE 8. Partition by $S_1 \oplus S_2$

We note that the end vertices of the edges common in S_1 and S_2 lie entirely either in V_5 or in V_6 but not in both. Therefore the ring sum of cut sets $S_1 \oplus S_2$ consists only of those edges that join vertices in V_5 to those in V_6.

Moreover, there are no edges outside $S_1 \oplus S_2$ which join vertices in V_5 with vertices of V_6.

Thus, the ringsum $S_1 \oplus S_2$ produces a partitioning of V into V_5 and V_6 such that

$$V_5 \cup V_6 = V \quad \text{and} \quad V_5 \cap V_6 = \emptyset$$

Hence, $S_1 \oplus S_2$ is a cut-set if the subgraphs containing V_5 and V_6 each remain connected after $S_1 \oplus S_2$ is removed from G.

Otherwise, $S_1 \oplus S_2$ is an edge disjoint union of cut-sets.

3.17. THEOREM. Prove that with respect to a given spanning tree T , a chord c_i that determines fundamental circuit τ, occurs in every fundamental cut-sets associated with the branches in τ and in no other cut-sets.

Proof : Consider a spanning tree T in a given connected graph G.
Let c_i be a chord with respect to T, and let the fundamental circuit determined by c_i be

$$\Gamma = \{c_i, b_1, b_2, \ldots, b_k\}$$

, which consists of k branches and the chord c_i.

Every branch of any spanning tree has a fundamental cut-set associated with it.
Let $S_1 = \{b_1, c_1, c_2, \ldots, c_q\}$ be the fundamental cut-set associated with b_1, consisting of q chords and the branch b_1

Now the edge b_1 is in Γ and S_1 both.
As there must be an even number of edges common in a cut-set and a circuit, there must be atleat one more edge common in Γ and S_1.

But among the remaining edges in Γ and S_1 only a chord can be common in both.

Therefore, the chord c_i is one of the chords $c_1, c_2, \ldots, c_q$.

Therefore, c_i is contained in the fundamental cut-set S_1 corresponding to the branch b_1 of Γ.

Exactly the same argument holds for fundamental cut-sets associated with $b_2, b_3, \ldots \ldots b_k$. Therefore, the chord c_i is contained in every fundamental cut-set associated with branches in Γ.

Moreover, if the chord c_i is in any other fundamental cut-set S' in T, besides those associated with $b_2, b_3, \ldots \ldots b_k$ then there would be only one edge c_i common to S' and Γ, since none of the branches in Γ are in S', .

But that is impossible as there must be an even number of edges common in a circuit and a cut-set.

Therefore c_i is contained only in the fundamental cut-sets corresponding to the branches of the fundamental circuit determined by c_i.

3.18. THEOREM. Prove that with respect to given spanning tree T , a branch b_i that determines fundamental cut-set S, is contained in every fundamental circuit associated with the chord in cut-set S and no other.

Proof : Consider a spanning tree T in a given connected graph G.
Let b_i be a branch with respect to T, and let the fundamental cut-set determined by b_i be

$$S = \{b_i, c_1, c_2, \ldots, c_p\}$$

, which consists of p chaords and the branch b_i.

Every chord of any spanning tree has a fundamental circuit associated with it.
Let $\Gamma_1 = \{c_1, b_1, b_2, \ldots, b_q\}$ be the fundamental circuit associated with c_1, consisting of q branches and the chord c_1

Now the edge c_1 is in Γ_1 and S both.
As there must be an even number of edges common in a cut-set and a circuit, there must be atleat one more edge common in Γ_1 and S.

But among the remaining edges in Γ_1 and S only a branch can be common in both.

Therefore, the branch b_i is one of the branches $b_1, b_2, \ldots, b_q$.

Therefore, b_i is contained in the fundamental circuit Γ_1 corresponding to the chord c_1 of S.

Exactly the same argument holds for fundamental circuits associated with $c_2, c_3, \ldots\ldots c_p$. Therefore, the branch b_i is contained in every fundamental circuit associated with chords in S.

Moreover, if the branch b_i is in any fundamental circuit Γ' in T, besides those associated with $c_2, c_3, \ldots\ldots c_p$ then there would be only one edge b_i common to Γ' and S, since none of the chords in S are in Γ', .

But that is impossible as there must be an even number of edges common in a circuit and a cut-set.

Therefore b_i is contained only in the fundamental circuits corresponding to the chords of the fundamental cut-set determined by b_i.

3.19. Definition. Edge Connectivity

The *Edge Connectivity* of a connected graph G is the number of edges in the smallest cut-set of G.
The edge connectivity of the graph in the figure 6 is 2.

3.20. Definition. Vertex Connectivity

The *Vertex Connectivity* of a connected graph G is the smallest number of vertices whose removal from the graph leaves the graph G disconnected.

3.21. Definition. Separable graph

A connected graph G is said to be *Seperable* if its vertex concnetivity is 1.

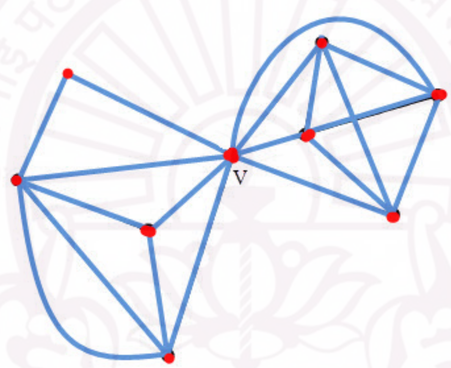


Figure 9

The graph in the figure 9 is a seperable graph as its vertex connectivity is 1.

3.22. Definition. Cut-Vertex

In a seperable graph a vertex whose removal from the graph leaves the graph disconnected is called a *Cut-Vertex* of the graph. The vertex V in of the graph in the figure 9 is a cut-vertex.

3.23. THEOREM. Prove that a vertex v in a connected graph G is a cut-vertex iff there exist two vertices x and y in G such that every path between x and y passes through v.

Proof : Let G be seperable graph with a cut-vertex v.
Removal of v from G results in a disconnected graph $G - v$.

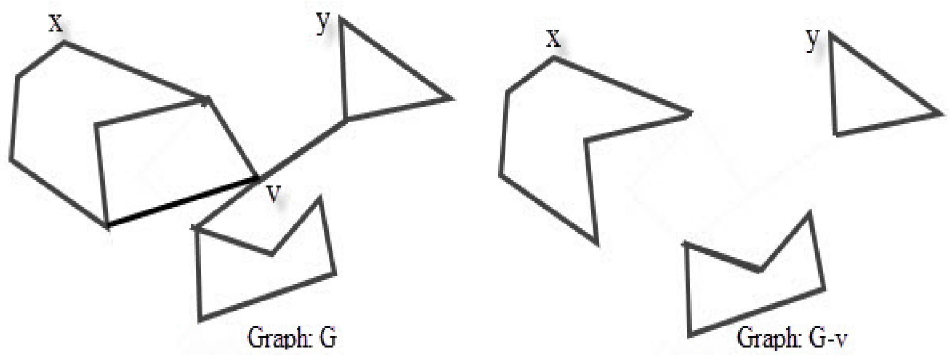


Figure 10

Suppose G consists of K components, say $G_1, G_2, \ldots, G_k$.
Choose some $x, y \in G - v$ such that $x \in G_i$ and $y \in G_j$ for some $i \neq j$.
There is no path between x and y as they belong to distinct components.
Since $G - v$ is a subgraph of G, we have $x, y \in G$.
As G is a connected graph there must be atleast one path between x and y in G.
As no path between x and y exist after removal of v form G, each path that connects x and y in G must be passing through v.

Conversely, suppose there are two vertices x and y in G such that every path joining them passes through a vertex v.
Let $P_1, P_2, \ldots, P_n$ be all possible paths connecting x and y.
As each path passes through v, on removing v from G there does not exist any path in $G - v$ that joins x and y.
Therefore $G - v$ is a disconnected graph.
As removal of v alone leaves G disconnected, v is a cut vertex of G.

3.24. THEOREM. Prove that the edge connectivity of a graph G can not exceed the degree of a vertex with the smallest degree in G.

Proof : Let G be a connected graph and v be a vertex of G with smallest degree.
Suppose $d(v) = k$
Therefore at the most k edges, that are incident with v, are required to be removed from G to isolate the vertex v.
Thus, removal of those k edges will leave the graph disconnected.
As the edge connectivity of a graph is the minimum number of edges required to removed from a graph to disconnect it, it follows that

$$\text{Edge connectivity of } G \leqslant k$$

Thus, the edge connectivity of a graph can not exceed the degree of a vertex with the smallest degree in the graph.

3.25. THEOREM. Prove that the vertex connectivity of any graph G can never exceed the edge connectivity of G.

Proof : Let G be a connected graph and the edge connectivity of G be α.

Therefore there exists a cut-set S of G containing α edges of G.
Suppose S partions the set of vertices of G into disjoint subsets V_1 and V_2

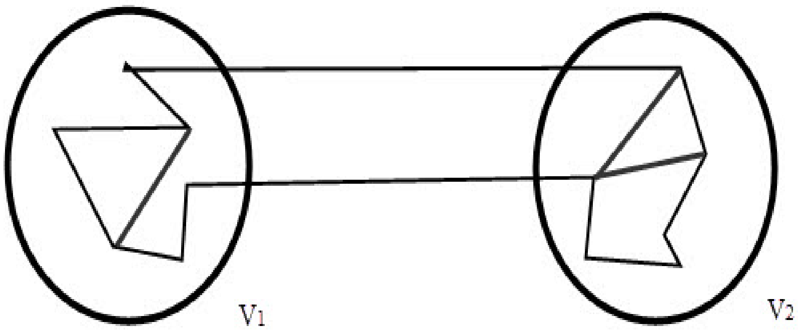


FIGURE 11

Therefore at the most α end-points of the edges in S lie in each of V_1 and V_2.
Thus, by removing at the most α vertices, that are incident with the edges in S, all the edges in S will be removed from G leaving it disconnected.
As the vertex connectivity of a connected graph is the smallest number of the vertices whose removal disconnects the graph, we have,

$$\text{vertex connectivity of } G \leqslant \alpha$$

Hence the theorem.

3.26. THEOREM. Prove that the maximum vertex connectivity one can achieve with a graph G of n vertices and e edges ($e \geqslant n-1$) is the integral part of the number $\frac{2e}{n}$.

Proof : Every edge in the graph G contributes two degrees to the total degrees of all n vertices.
As e edges contribute total $2e$ degrees, there are in all $2e$ degrees divided among n vertices.
The least integer not exceeding the average of degree of vertices $\frac{2e}{n}$ is $\left[\frac{2e}{n}\right]$
Therefore, there must be at least one vertex in G whose degree is less than or equal to $\left[\frac{2e}{n}\right]$.
We know that the vertex connectivity of a graph does not exceed degree of any vertex in a connected graph.
Therefore the vertex connectivity of G does not exceed $\left[\frac{2e}{n}\right]$.

3.27. DEFINITION. Define k-connected graph

A connected graph G is called a k-connected graph if the vertex connectivity of G is k.

3.28. Example. Describe network flows

Network Flows
In real life many times it is very important to know the maximum rate of flow that is possible from one station to another station in certain networks like telephone lines, highways, railroads, pipelines carring gas or oil or water. In such networks the flow depends on the individual capacity of the lines joining stations, like roads, pipelines etc. Such networks are represented by weighted connected graphs in which the vertices are stations and the edges are lines. The weight, a real positive number, associated with edge represents the capacity of the line , that is , the maximim amount of flow possible per unit of time.

Through graphical study of such networks we try to find answer of questions like

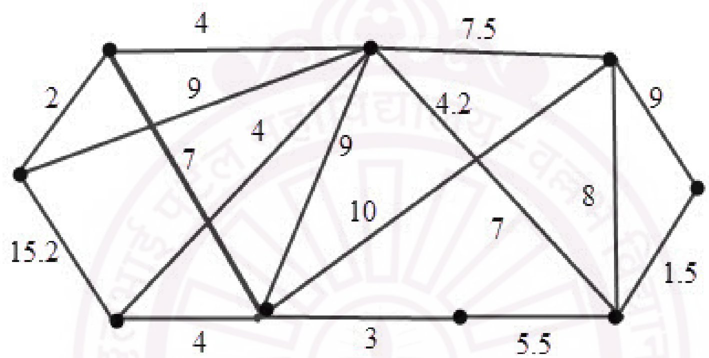


Figure 12

(i) What is the maximum flow possible through the network between a specified pair of vertices and
(ii) how to achieve this flow?
Thus, graph theory helps us solve network flow problems.