CHAPTER #02

INDETERMINATE FORMS

CONICS

EXAMPLE #01

Evaluate $\lim_{x\to 0}\frac{\log(1-x)+1+\sin x-\cos x}{x\tan^2 x}$

SOLUTION $\lim_{x\to 0}\frac{\log(1-x)+1+\sin x-\cos x}{x\tan^2 x}=\lim_{x\to 0}\frac{\log(1-x)+1+\sin x-\cos x}{x\frac{\tan^2 x}{x^2}\cdot x^2}$

$$=\lim_{x\to 0}\frac{\log(1-x)+1+\sin x-\cos x}{x^3}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{(1-x)}+\cos x+\sin x}{3x^2}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{(1-x)^2}-\sin x+\cos x}{6x}$$

$$=\lim_{x\to 0}\frac{-\frac{2}{(1-x)^3}-\cos x-\sin x}{6}$$

$$=\frac{-2-1}{6}=-\frac{1}{2}.$$

$$\therefore \lim_{x\to 0}\frac{\log(1-x)+1+\sin x-\cos x}{x\tan^2 x}=-\frac{1}{2}$$

EXAMPLE#02

If $\lim_{x\to 0}\frac{a\sin x-bx+cx^2+x^3}{2x^2\log(1+x)-2x^3+x^4}=\frac{3}{40}$, then find the ,then find the values of the constants a,b,c

Solution The given limit is of the form $\frac{0}{0}$. And the limit is $\frac{3}{40}$

$$\frac{3}{40}=\frac{1}{2}\lim_{x\to 0}\frac{a\cos x-b+2cx+3x^2}{2x\log(1+x)+\frac{x^2}{1+x}-3x^2+2x^3}$$

Now, as x tends to 0, the denominator tends to zero and the given limit is $\frac{3}{40}$, the numerator must tend to zero.

i.e., $a-b=0$. _____(1)

$$\therefore\ \frac{3}{40}=\frac{1}{2}\lim_{x\to 0}\frac{-a\sin x+2c+6x}{2\log(1+x)+\frac{2x}{1+x}+\frac{(2x+x^2)}{(1+x)^2}-6x+6x^2}$$ (by L'Hospital's Rule1)

Again, as x tends to 0, the denominator tends to zero and the given limit is $\frac{3}{40}$, the numerator must tend to zero.

$$i.e., \ c = 0.$$

$$\therefore \ \frac{3}{40} = \frac{1}{2}\lim_{x\to 0}\frac{-a\sin x + 6x}{2\log(1+x) + \frac{2x}{1+x} + \frac{(2x+x^2)}{(1+x)^2} - 6x + 6x^2} \quad \left(\text{form } \tfrac{0}{0}\right)$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{-a\cos x + 6}{\frac{2}{1+x} + \frac{2}{(1+x)^2} + \frac{2}{(1+x)^3} - 6 + 12x} \quad \text{(by L'Hospital's Rule1)}$$

Again, as x tends to 0, the denominator tends to zero and the given limit is $\frac{3}{40}$, the numerator must tend to zero.

$$i.e., \ a = 6.$$

And by equation (1), $b = a$. Therefore, $a = 6, b = 6, c = 0$.

EXAMPLE #03

Evaluate $\lim_{x\to 0}\frac{\log(\log(1-3x^2))}{\log(\log(\cos 2x))}$.

Solution

Let $L=\lim_{x\to 0}\frac{\log(\log(1-3x^2))}{\log(\log(\cos 2x))}$. Then the given limit is of the indeterminate form $\frac{\infty}{\infty}$

$$=\lim_{x\to 0}\frac{\frac{1}{\log(1-3x^2)}\cdot\frac{1}{(1-3x^2)}\cdot(-6x)}{\frac{1}{\log(\cos 2x)}\cdot\frac{1}{\cos 2x}\cdot(-2\sin 2x)}$$

$$=3\lim_{x\to 0}\frac{\log(\cos 2x)}{\log(1-3x^2)}\cdot\frac{\cos 2x}{1-3x^2}\cdot\frac{x}{\sin 2x}$$

$$=\frac{3}{2}\lim_{x\to 0}\frac{\log(\cos 2x)}{\log(1-3x^2)}\cdot\lim_{x\to 0}\frac{\cos 2x}{1-3x^2}\cdot\lim_{x\to 0}\frac{2x}{\sin 2x}$$

$$= \frac{3}{2} \lim_{x \to 0} \frac{\log(\cos 2x)}{\log(1 - 3x^2)} \cdot \frac{1}{1 - 0}$$

$$= \frac{3}{2} \lim_{x \to 0} \frac{\log(\cos 2x)}{\log(1 - 3x^2)}$$

$$= \frac{3}{2} \lim_{x \to 0} \frac{\frac{1}{\cos 2x} \cdot (-2 \sin 2x)}{\frac{1}{1 - 3x^2}(-6x)}$$ (l

$$= \frac{3}{2} \lim_{x \to 0} \frac{1 - 3x^2}{\cos 2x} \cdot \frac{\sin 2x}{3x}$$

$$= \frac{\cancel{3}}{2} \lim_{x \to 0} \frac{1 - 3x^2}{\cos 2x} \cdot \lim_{x \to 0} \frac{\sin 2x}{\cancel{3} x} = 1.$$

EXAMPLE #04

Evaluate $\lim_{x\to a}\left(x^2-a^2\right)\tan\left(\frac{\pi x}{2a}\right)$, $a\neq 0$

Solution Let $L=\lim_{x\to a}\left(x^2-a^2\right)\tan\left(\frac{\pi x}{2a}\right)$. Then the given limit is of the indeterminate form $0\times\infty$

$$L=\lim_{x\to a}\frac{x^2-a^2}{\cot\left(\frac{\pi x}{2a}\right)}$$

$$=\lim_{x\to a}\frac{2x}{-\frac{\pi}{2a}\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)}$$

$$=-\frac{4a}{\pi}\cdot\frac{a}{\operatorname{cosec}^2\frac{\pi}{2}}$$

$$=-\frac{4a^2}{\pi}.$$

EXAMPLE #05

Evaluate $\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\log x}\right]$

Solution Let $L=\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\log x}\right]$. Then given limit is of the form ∞ -∞

$$L=\lim_{x\to 1}\frac{x\log x-x+1}{(x-1)\log x} \qquad \left(\text{form } \tfrac{0}{0}\right)$$

$$=\lim_{x\to 1}\frac{x\log x-x+1}{x\log x-\log x} \qquad \left(\text{form } \tfrac{0}{0}\right)$$

$$=\lim_{x\to 1}\frac{\log x}{1+\log x-\frac{1}{x}} \qquad \left(\text{form } \tfrac{0}{0}\right)$$

$$=\lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} \qquad \text{(byL'Hospital's Rule 1)}$$

$$=\frac{1}{2}.$$

EXAMPLE #06

Evaluate $\lim_{x\to\frac{\pi}{2}} (\cos x)^{\cos^2 x}$

Solution

Let $L = \lim_{x\to\frac{\pi}{2}} (\cos x)^{\cos^2 x}$ .Then given limit is of the form 0^0

Taking log on both sides, we get

$$\log L = \lim_{x\to\frac{\pi}{2}} \cos^2 x \log(\cos x) \qquad (\text{form } 0\times\infty)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\log(\cos x)}{\sec^2 x} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-\tan x}{2\sec^2 x \tan x}$$

$$= 0 \quad \Rightarrow L = e^0 = 1.$$

EXAMPLE #07

Find the focus and directrix of the parabola $x^2 = -8y$. Then sketch it.

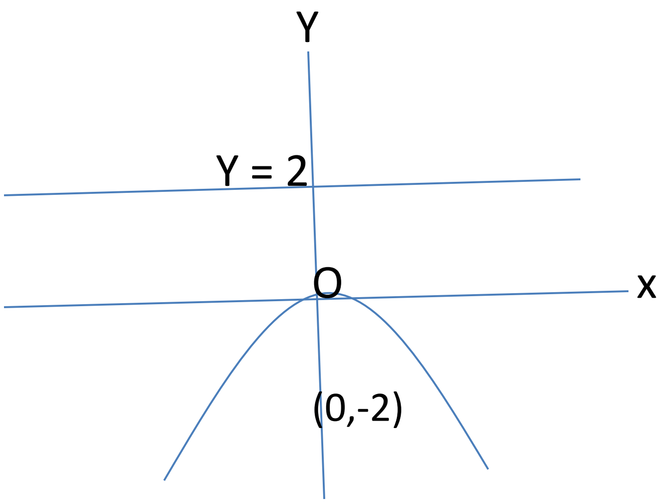
Solution

Comparing with $x^2 = -4py$,we get

$4p = 8 \Rightarrow p = 2$

Focus : $(0, -p) = (0,-2)$

Directrix : $y = p \Rightarrow y = 2$



EXAMPLE #08

Find semi major axis , semi minor axis, centre to focus distance, foci, directrix , vertices, eccentricity and centre for $169x^2 + 25y^2 = 4225$

Solution

Given that

$169x^2 + 25y^2 = 4225$

Dividing by 4225 both sides, we get

$$\frac{x^2}{25} + \frac{y^2}{169} = 1$$

Semi major axis: a = 13

Semi minor axis : b = 5

$$c = \sqrt{a^2 - b^2} = \sqrt{169 - 25} = \sqrt{144} = 12$$

Foci : (0, ± c) = (0,±12)

Vertices : (0, ± a) = (0, ± 13)

Centre : (0,0)

Eccentricity = c\a = 12\13

Directrix y = a\e = 169\12 = ± 14.08

EXAMPLE #09

Find the equation of hyperbola centre at origin given that foci : $(0, \pm\sqrt{2})$, Asymptote : $y = \pm x$

Solution

Foci :$(0, \pm\sqrt{2}) = (0, \pm c) \Rightarrow c = \sqrt{2}$

Asymptote : $y = \pm \frac{a}{b} x$

Given that $y = \pm x \Rightarrow a\backslash b = 1 \Rightarrow a = b$

We know that $c^2 = a^2 + b^2 \Rightarrow 2 = 2a^2 \Rightarrow a = 1$

Hence required equation of hyperbola is

$x^2 - y^2 = 1$

EXAMPLE #10

Use discriminant $B^2 - 4AC$ to decide whether the following equations represents parabola,ellipse or hyperbola

(i) $2x^2 - 8xy + 8y^2 + 2x + 2y = 0$

(ii) $x^2 + 4xy + 4y^2 - 3x = 6$

(iii) $xy + y^2 - 3x = 5$

(iv) $3x^2 - 18xy + 27y^2 - 5x + 7y = -4$

Solution The quadratic curve represented by equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is

(i) a **parabola** if $B^2 - 4AC = 0$

(ii) an **ellipse** if $B^2 - 4AC < 0$

(iii) a hyperbola if $B^2 - 4AC > 0$

(i) Comparing eq $2x^2 - 8xy + 8y^2 + 2x + 2y = 0$ with $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ we get $A = 2$, $B = -8$, $C = 8$

$B^2 - 4AC = 0$, Hence it represents parabola

(ii) Comparing eq $x^2 + 4xy + 4y^2 - 3x = 6$ with $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, we get $A = 1$, $B = 4$, $C = 4$.Therefore, $B^2 - 4AC = 0$. Hence it represents parabola

(iii) Comparing eq $xy + y^2 - 3x = 5$ with $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,we get $A = 0$, $B = 1$, $C = 1$. Therefore, $B^2 - 4AC > 0$. Hence it represents hyperbola.

(iv) Comparing eq $3x^2 - 18xy + 27y^2 - 5x + 7y = -4$ with $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ $A = 3$, $B = -18$, $C = 27$.Therefore, $B^2 - 4AC = 0$. Hence it represents parabola