CHAPTER #01

HYPERBOLIC FUNCTIONS,

HIGHER ORDER DERIVATIVES,

LEIBNITZ THEOREM,

TAYLOR'S THEOREM

Example #01

If y = log(tanx),prove that

(i) 2coshny cosec2x = cosh(n+1)y + cosh(n – 1)y

(ii)2sinhny cot2x = cosh(n - 1)y – cosh(n +1)y

Solution y = log(tanx) therefore $\tan x = e^{y} and \cot x = e^{-y}$

(i) cosh(n+1)y + cosh(n – 1)y =

$$= \frac{e^{(n+1)y} + e^{-(n+1)y}}{2} + \frac{e^{(n-1)y} + e^{-(n-1)y}}{2}$$

$$= \frac{1}{2}\left[e^{ny}e^{y} + e^{-ny}e^{-y} + e^{ny}e^{-y} + e^{-ny}e^{y}\right]$$

$$= \frac{1}{2}\left[e^{ny}\left(e^{y} + e^{-y}\right) + e^{-ny}\left(e^{y} + e^{-y}\right)\right]$$

$$= \frac{1}{2}\left(e^{ny} + e^{-ny}\right)\left(e^{y} + e^{-y}\right)$$

$$= 2\left(\frac{e^{ny} + e^{-ny}}{2}\right)\left(\frac{e^{y} + e^{-y}}{2}\right)$$

$$= 2\cosh ny\left(\frac{e^{y} + e^{-y}}{2}\right)$$

$$= 2\cosh ny\left(\frac{\tan x + \cot x}{2}\right)$$

$$= 2\cosh ny\left[\frac{1}{2}\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)\right]$$

$$= 2\cosh ny\left[\frac{1}{2}\left(\frac{\sin^2 x + \cos^2 x}{\sin x\cos x}\right)\right]$$

$$= 2\cosh ny\frac{1}{2\sin x\cos x}$$

$$= 2\cosh ny\cos ec2x$$

(ii) $\cosh(n-1)y - \cosh(n+1)y$

$$= \frac{1}{2}\left[e^{ny}e^{-y} + e^{-ny}e^{y} - e^{ny}e^{y} - e^{-ny}e^{-y}\right]$$

$$= \frac{1}{2}\left[-e^{ny}\left(e^{y} - e^{-y}\right) + e^{-ny}\left(e^{y} - e^{-y}\right)\right]$$

$$= 2\left[-\frac{e^{ny} - e^{-ny}}{2}\right]\left[\frac{e^{y} - e^{-y}}{2}\right]$$

$$= -2\sinh ny\left(\frac{\tan x - \cot x}{2}\right)$$ (I

$$= -2\sinh ny\left[\frac{\sin^2 x - \cos^2 x}{2\sin x\cos x}\right]$$

$$= 2\sinh ny\left[\frac{\cos^2 x - \sin^2 x}{\sin 2x}\right]$$

$$= 2\sinh ny\cot 2x.$$

Example#02

Prove that $\cosh^2 x = \cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{1-\cosh^2 x}}}$.

Solution

$$R.H.S. = \cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{1-\cosh^2 x}}} = \cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{-\sinh^2 x}}}$$

$$= \cfrac{1}{1-\cfrac{1}{1+\operatorname{cosech}^2 x}}$$

$$= \cfrac{1}{1-\cfrac{1}{\coth^2 x}}$$

$$= \frac{1}{1-\tanh^2 x}$$

$$= \frac{1}{\operatorname{sech}^2 x} \qquad = \cosh^2 x$$

Example#3

For real z prove that $\sinh^{-1} z = \log\left(z + \sqrt{z^2+1}\right)$

Solution

Let $\sinh^{-1} z = x$. Then $\sinh x = z$.

$$\Rightarrow z = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x}-1}{2e^x}$$

$$\Rightarrow e^{2x} - 2z\,e^x - 1 = 0,$$

Which is a quadratic equation in e^z

Therefore, $e^x = \dfrac{2z \pm \sqrt{4z^2+4}}{2} = z \pm \sqrt{z^2+1}$.

By taking positive sign,

$$e^x = z + \sqrt{z^2+1}$$

$$\Rightarrow x = \log\left(z + \sqrt{z^2+1}\right).$$

Therefore, $\sinh^{-1} z = \log\left(z + \sqrt{z^2+1}\right)$.

Example#4

Prove that $\cosh^{-1}\left(\sqrt{1+x^2}\right)=\tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

Solution Let $\cosh^{-1}\left(\sqrt{1+x^2}\right)=y$. Then $\sqrt{1+x^2}=\cosh y$

$$\Rightarrow 1+x^2=\cosh^2 y$$

$$\Rightarrow x^2=\cosh^2 y-1=\sinh^2 y$$

$$\Rightarrow x=\sinh y$$

$$\Rightarrow \frac{\sinh y}{\cosh y}=\frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow \tanh y=\frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow y=\tanh^{-1}\left[\frac{x}{\sqrt{1+x^2}}\right].$$

Hence, $\cosh^{-1}\left(\sqrt{1+x^2}\right)=\tanh^{-1}\left[\frac{x}{\sqrt{1+x^2}}\right]$.

Example#5

Find n[th] order derivative of $y=\frac{1}{x^3+6x^2+11x+6}$

Solution

$$x^3+6x^2+11x+6=x^2(x+1)+5x(x+1)+6(x+1)$$
$$=(x+1)(x^2+5x+6)$$
$$=(x+1)(x+2)(x+3).$$

$$\therefore\ y=\frac{1}{(x+1)(x+2)(x+3)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}+\frac{C}{(x+3)}$$

$$\Rightarrow 1=A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)$$

Putting $x=-1,-2,-3$; we have $A=\frac{1}{2}$, $B=-1$, $C=\frac{1}{2}$ respectively.

$$\therefore\ y=\frac{1}{(x+1)(x+2)(x+3)}=\frac{1}{2(x+1)}-\frac{1}{x+2}+\frac{1}{2(x+3)}.$$

Differentiating n times , we get $y_n=\frac{(-1)^n n!}{2(x+1)^{n+1}}-\frac{(-1)^n n!}{(x+2)^{n+1}}+\frac{(-1)^n n!}{2(x+3)^{n+1}}$

$$=\frac{(-1)^n n!}{2}\left[\frac{1}{(x+1)^{n+1}}-\frac{2}{(x+2)^{n+1}}+\frac{1}{(x+3)^{n+1}}\right].$$

Example#6

Find n[th] order derivative of $y = e^x \sin^4 x$

Solution Here $y = e^x \sin^4 x = e^x (\sin^2 x)^2 = e^x \left[\dfrac{1-\cos 2x}{2}\right]^2$

$$= \frac{1}{4} e^x \left[1 - 2\cos 2x + \cos^2 2x\right]$$

$$= \frac{1}{4} e^x \left[1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right]$$

$$= \frac{1}{8} e^x \left[3 - 4\cos 2x + \cos 4x\right]$$

$$= \frac{1}{8}\left[3e^x - 4e^x \cos 2x + e^x \cos 4x\right].$$

Differentiating n times , we get $y_n = \frac{1}{8}\Big[3 \cdot 1^n e^x - 4e^x (1+4)^{\frac{n}{2}} \cos\left(2x + n\tan^{-1}\left(\tfrac{2}{1}\right)\right)$

$$+ e^x (1+16)^{\frac{n}{2}} \cos\left(4x + n\tan^{-1}\left(\tfrac{4}{1}\right)\right)\Big]$$

$$= \frac{e^x}{8}\Big[3 - 4 \cdot 5^{\frac{n}{2}} \cos\left(2x + n\tan^{-1}(2)\right)$$

$$+ 17^{\frac{n}{2}} \cos\left(4x + n\tan^{-1}(4)\right)\Big].$$

Example#7

Find the n[th] order derivative of $y = x^2 \log(3x + 5)$

Solution Let $u = \log(3x+5),\ v = x^2$ ....(1)

Now differentiating equation (1) n times with respect to x and applying Leibniz's Rule, we have,

$$y_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \frac{n(n-1)(n-2)}{3!} u_{n-3} v_3 + \cdots \cdots + u v_n$$

$$= \frac{3^n (-1)^{n-1} (n-1)!}{(3x+5)^n} x^2 + n \frac{3^{n-1} (-1)^{n-2} (n-2)!}{(3x+5)^{n-1}} (2x)$$

$$+ \frac{n(n-1)}{2} \cdot \frac{3^{n-2} (-1)^{n-3} (n-3)!}{(3x+5)^{n-2}} (2) + 0$$

$$= \frac{3^n (-1)^{n-1} (n-1)!}{(3x+5)^n} x^2 + 2n x \frac{3^{n-1} (-1)^{n-2} (n-2)!}{(3x+5)^{n-1}}$$

$$+ \frac{n(n-1) 3^{n-2} (-1)^{n-3} (n-3)!}{(3x+5)^{n-2}}.$$

Example#8

If $y=\sin^{-1}x$ (or $x=\sin y$), prove that

$$(i)\left(1-x^2\right)y_2-xy_1=0,$$

$$(ii)\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0.$$

Solution Here, $y = \sin^{-1} x$

Differentiating with respect to x, we get

$$y_1=\frac{1}{\sqrt{1-x^2}}.$$

Squaring both sides, we have,

$$\left(1-x^2\right)y_1^2=1.$$

Again differentiating with respect to x, we get,

$$\left(1-x^2\right)2y_1y_2+y_1^{\,2}\left(-2x\right)=0.$$

Dividing both sides by $2y_1$,

$$\left(1-x^2\right)y_2-xy_1=0.$$

Now differentiating n times with respect to x and applying Leibniz's Rule, we get,

$$\left(1-x^2\right)y_{n+2}+n\left(-2x\right)y_{n+1}+\frac{n\left(n-1\right)}{\not{2}}\left(-\not{2}\right)y_n$$

$$+0+0+\cdots\ \cdots+0-xy_{n+1}-n\left(1\right)y_n-0-0-\cdots-0=0.$$

$$\Rightarrow\left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-n^2y_n=0.$$

Example#9

Find the Maclaurin's series expasion for the function f(x)=cosx

Solution Let $f(x) = \cos x$. Then $f(0) = \cos 0 = 1$

Differentiating n times w.r.t. x and using the standard result,

$$f^{(n)}(x) = \cos\left(x + \tfrac{n\pi}{2}\right) \quad \text{and} \quad f^{(n)}(0) = \cos\left(\tfrac{n\pi}{2}\right)$$

Now for $n = 1, 2, 3, 4, 5, \ldots$, we get,

$$f^{(1)}(0) = \cos\tfrac{\pi}{2} = 0, \quad f^{(2)}(0) = \cos\pi = -1, \quad f^{(3)}(0) = \cos\tfrac{3\pi}{2} =$$

$$f^{(4)}(0) = \cos 2\pi = 1, \ \ldots \text{ respectively.}$$

Thus, the Maclaurin's series of $f(x) = \cos x$ is,

$$\cos x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots\cdots$$

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots$$

Example#10

Find Maclaurin's series of f(x) = coshx

Solution As we know that $\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \frac{x^{4}}{4!} + \frac{x^{5}}{5!} + \frac{x^{6}}{6!} + \cdots\cdots.$$

$$\therefore e^{-x} = 1 + \frac{(-x)}{1!} + \frac{(-x)^{2}}{2!} + \frac{(-x)^{3}}{3!} + \frac{(-x)^{4}}{4!} + \frac{(-x)^{5}}{5!} + \frac{(-x)^{6}}{6!} + \cdots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^{2}}{2!} - \frac{x^{3}}{3!} + \frac{x^{4}}{4!} - \frac{x^{5}}{5!} + \frac{x^{6}}{6!} - \cdots\cdots.$$

$$\therefore e^{x} + e^{-x} = 2\left[1 + \frac{x^{2}}{2!} + \frac{x^{4}}{4!} + \frac{x^{6}}{6!} + \cdots\cdots\right].$$

$$\therefore \cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right) = 1 + \frac{x^{2}}{2!} + \frac{x^{4}}{4!} + \frac{x^{6}}{6!} + \cdots\cdots$$